NECESSITIES GENERATED BY AN INITIAL VALUATION

by

Loredana Biacino* and Giangiacomo Gerla**

1. Introduction.

In this paper we propose a technique to generate a necessity given an initial valuation of the events: this technique is more general and substantially different from Shafer's building of consonant belief functions upon a probability mass distribution in that the former is of the lattice theoretical kind, while the latter employs the additive structure of the real number interval [0,1]. Because of its lattice nature, our technique could be applied even to necessities and possibilities which eventually took values in a complete lattice. We do not require the null event is valued zero; so, the arising class of necessities (and possibilities as well) turn out a complete lattice, namely a closure system.

We assume that the necessities are defined in a Boolean algebra. As an example, such an algebra can be the algebra of the subsets of a given set, the Lindenbaum algebra of the sentences of a given first order language, an so on.

This paper will be continued in a next paper in this Journal. The proofs of all the propositions can be found in Biacino and Gerla [1990].

2. Preliminaries.

In the sequel B denotes a Boolean algebra whose elements are called *events*, we denote by 0 and 1 the minimum and the maximum, respectively. The class F(B) of the maps from B to [0,1] is a complete lattice with respect to the operations $\wedge$ and $\vee$ defined by

$$(\wedge s_i)(x)=\inf\{s_i(x)/x\in B\} \quad ; \quad (\vee s_i)(x)=\sup\{s_i(x)/x\in B\}$$

where $(s_i)_{i\in I}$ is any family of elements of F(B). If $\alpha\in[0,1]$, then the subset $C(s,\alpha) = \{x\in B/s(x)\geq\alpha\}$ is called the *α-cut* of s. We say that an element n of

F(B) is a *necessity* if

(2.1) $$n(x\wedge y)=n(x)\wedge n(y) \quad \text{and} \quad n(1)=1,$$

and we denote by N(B) the set of the necessities defined on B. We call *degree of contradictoriness* the number n(0) and we denote it by $C_r(n)$; since n is increasing, $C_r(n)$ is the minimum of n. We say that n is *completely consistent* if $C_r(n)=0$ an that n is *completely inconsistent* if $C_r(n)=1$. Obviously a completely inconsistent necessity is constantly equal to 1 and gives no information about the events.

We say that an element p of F(B) is a *possibility* if

(2.2) $$p(x\vee y)=p(x)\vee p(y) \quad \text{and} \quad p(0)=0.$$

We denote by P(B) the class of possibilities defined on B; we call *degree of consistence* the maximum p(1), and denote it by $C_s(p)$. Moreover, we say that p is *completely consistent* if $C_s(p)=1$ and *completely inconsistent* if $C_s(p)=0$.

The following propositions are obvious extensions of well known results (see D.Dubois and H.Prade [1988]).

Proposition 2.1 If n is a necessity then, for every $x,y\in B$,

a) either $n(x)=C_r(n)$ or $n(-x)=C_r(n)$;

b) $n(x\vee y)\geq n(x)+n(y)-n(x\wedge y)$.

If p is a possibility, then

c) either $p(x)=C_s(p)$ or $p(-x)=C_s(p)$

d) $p(x\vee y)\leq p(x)+p(y)-p(x\wedge y)$.

In the sequel, given an element f of F(B), $\sim f$ is defined by $\sim f(x)=1-f(-x)$. The operation $\sim$ fulfills the following properties

(2.3) $$\sim(\sim f)=f, \quad f\leq g \Rightarrow \sim f\geq\sim g, \quad \sim(\bigvee f_i)=\bigwedge(\sim f_i), \quad \sim(\bigwedge f_i)=\bigvee(\sim f_i)$$

where $f,g\in F(B)$ and $\langle f_i\rangle$ is any family of elements of F(B).

Proposition 2.2 For every element f of F(B)

a) f necessity $\Rightarrow$ $f \leq \sim f + C_r(f)$ (i.e. $f(x)+f(-x) \leq 1 + C_r(f)$)

b) f possibility $\Rightarrow$ $f \geq \sim f - 1 + C_s(f)$ (i.e. $f(x)+f(-x) \geq C_s(f)$)

c) f completely consistent necessity $\Rightarrow$ $f \leq \sim f$

d) f completely consistent possibility $\Rightarrow$ $f \geq \sim f$.

e) f possibility $\Longleftrightarrow$ $\sim f$ necessity.

We say that a map $f:B \rightarrow \{0,1\}$ is a *Boolean valuation* if is a homomorphism from B into the Boolean algebra {0,1}. This means that $f(x \wedge y) = f(x) \wedge f(y)$, $f(x \vee y) = f(x) \vee f(y)$ and $f(-x) = \sim f(x)$ for every $x,y \in B$ and therefore a classical valuation is both a necessity and a possibility. Conversely, the following proposition holds.

Proposition 2.3 For every function $f:B \rightarrow [0,1]$ the following are equivalent

a) f is both a necessity and a possibility ;

b) f is a (characteristic function of a) prime filter of B ;

c) f is a Boolean valuation .

Since the concept of possibility is dual of the concept of necessity, we limit ourselves to examine the necessities.

The following proposition gives some obvious characterizations of the necessities.

Proposition 2.4 Let n be an element of F(B) such that n(1)=1, then the following propositions are equivalent:

a) n is a necessity;

b) n is increasing and $n(x \wedge y) \geq n(x) \wedge n(y)$;

c) $n(x \wedge y) \geq n(x) \wedge n(y)$ and $n(x \vee y) \geq n(x) \vee n(y)$;

d) n is closed with respect to Modus Ponens, i.e. $n(y) \geq n(x \rightarrow y) \wedge n(x)$;

e) every cut of n is a filter of B.

From e) of Proposition 2.4 it follows that the filters of B are necessities.

Since the filters in a Lindenbaum Boolean algebra coincide with the theories, the necessities can be viewed as a generalization of the notion of theory in the first order logic. This suggests the following considerations. Let T be a theory and α a formula, then $\alpha \notin T$ does not mean that α is false but, in a sense, that we have not sufficient information in order to prove α. The theory T expresses the falsity of α only if the negation $-\alpha$ of α belongs to T. Analogously, if n is a necessity, then $n(\alpha)=0$ does not mean that, in our opinion, α is false but that we have not enough information in order to support our belief in α. As a matter of fact, the information about the falsity of α is expressed by $n(-\alpha)$; it could happen even that both $n(\alpha)$ and $n(-\alpha)$ are equal to zero. Dual considerations hold for the possibilities. Indeed, since a possibility p is equal to the dual $\sim n$ of a necessity, the equality $p(\alpha)=1$ is equivalent to $n(-\alpha)=0$ and means that we have no reason to believe α false.

The following proposition shows that the necessities can be identified with suitable families of filters.

Proposition 2.5 The necessities can be identified with the families $\langle C_\alpha \rangle_{\alpha \in I}$ of filters of B with I complete subset of (0,1] and

(1.4) $$\bigcap C_{\alpha_i} = C_{\bigvee \alpha_i}\ .$$

for every family $\langle \alpha_i \rangle$ of elements of I.

As an example, any finite chain of filters

$$F_{\alpha_1} \supset \ldots \supset F_{\alpha_m} \quad \text{with} \quad 0 \le \alpha_1 \le \ldots \le \alpha_m \le 1$$

defines a necessity.

3. Generated necessities.

In the sequel an *initial valuation* is any map defined in a subset D_f of B and with values in [0,1]. The elements of D_f are sometime called *focal events* of f. We denote by $\sim f$ the initial valuation such that

$D_{\sim f}=\{x\in B/-x\in D_f\}$ and $\sim f(x)=1-f(-x)$. If f and g are two initial valuations we set $f\le g$ provided that $f(x)\le g(x)$ for every $x\in D_f\cap D_g$. The join $f\vee g$ is defined on $D_f\cup D_g$ by

$$(3.1)\qquad (f\vee g)(x)=\begin{cases} f(x)\vee g(x) & \text{if } x\in D_f\cap D_g \\ f(x) & \text{if } x\in D_f-D_g \\ g(x) & \text{if } x\in D_g-D_f\,. \end{cases}$$

Putting in (3.1) $f(x)\wedge g(x)$ in place of $f(x)\vee g(x)$ we obtain the definition of the initial valuation $f\wedge g$. The properties given in (2.3) can be easily extended to the initial valuations.
In this section we examine the question of generating a necessity and a possibility in accordance with an initial valuation.

Proposition 3.1 The meet $\bigwedge n_i$ of a family $\langle n_i\rangle$ of necessities is a necessity. If f is an initial valuation then $\bar{f}=\bigwedge\{g\in N(B)/g\ge f\}$ can be obtained by

$$(3.2)\quad \bar{f}(z)=\begin{cases} 1 & \text{if } z=1 \\ \bigvee\{f(y_1)\wedge\ldots\wedge f(y_m)\,/\,y_1\wedge\ldots\wedge y_m\le z \text{ and } y_i\in D_f\} & \text{if } z\ne 1. \end{cases}$$

We say that $\bar{f}$ is the necessity *generated* by the initial valuation f: in a sense $\bar{f}$ can be viewed as the "theory" generated by the "system of axioms " f. Notice that in the class of completely consistent necessities the operator $^{-}$ is not always defined; indeed it is possible that no completely consistent necessity is greater than f. This is the main reason for which we have skipped the condition $n(0)=0$ in defining the necessities. We call *degree of contradictoriness* $C_r(f)$ of f the degree of contradictoriness $C_r(\bar{f})$ of $\bar{f}$, and, obviously, $C_r(f)=\bigvee\{f(y_1)\wedge\ldots\wedge f(y_n)\,/\,y_1\wedge\ldots\wedge y_n=0$ and $y_i\in D_f\}$.

Notice that the events x such that $f(x)=0$ have no influence in determining the necessity $\bar{f}$. Consequently, it is not restrictive to assume that an initial valuation is constantly different from zero in its domain. This is in accordance with the fact that $f(x)=0$ means that we have no opinion on the event x ; not that we think that x is false.

It is very natural to assume that the initial valuation f is finite, i.e. it is addressed only to a finite number of focal events, $D_f=\{e_1, \dots, e_n\}$. In this case (3.2) defines $\bar{f}$ in a constructive simple way, obviously.

<u>Example 3.1</u> The initial valuation f is defined in one element only ; $D_f=\{e\}$. Then, if $f(e)=\alpha$, f generates the necessity n^e_α defined by

$$n^e_\alpha(x)=\begin{cases}1 & \text{if } x=1\ ;\\ \alpha & \text{if } x\geq e\\ 0 & \text{otherwise.}\end{cases}$$

In the case $\alpha=1$, n^e_1 is the characteristic function of the principal filter generated by e and will be denoted by n^e.

<u>Example 3.2</u> Only two focal events e_1 and e_2 are considered, $f(e_1)=1/2$ and $f(e_2)=1$. Thus

$$\bar{f}(x)=\begin{cases}1 & \text{if} \quad x\geq e_2\\ 0 & \text{if} \quad x\not\geq e_1\cap e_2\\ 1/2 & \text{if} \quad x\geq e_1\cap e_2 \text{ and } x\not\geq e_2\end{cases}$$

and $C_r(f)=1/2$ if $e_1\cap e_2=0$ while $C_r(f)=0$ otherwise.

The dual of Proposition 3.1 holds.

Proposition 3.4 The join $\bigvee p_i$ of a family $\langle p_i\rangle$ of possibilities is a possibility. In particular, if f is an initial valuation then $\mathring{f}=\bigvee\{g\in P(B)/g\leq f\}$ is a possibility and $\mathring{f}$ can be defined by

$$(3.3)\qquad \mathring{f}(z)=\begin{cases}0 & \text{if } z=0\\ \bigwedge\{f(y_1)\vee\dots\vee f(y_n)\ /\ y_1\vee\dots\vee y_n\geq z,\ y_1,\dots,y_n\in D_f\} & \text{if } z\neq 0.\end{cases}$$

We say that $\mathring{f}$ is the *possibility generated by f*, we set $C_s(f)=C_s(\mathring{f})$ and we say that $C_s(f)$ is the *degree of consistence* of f. Obviously, initial valuations should be taken keeping in mind if we want to evaluate the possibility or the necessity of the event under consideration. For instance, the events x such that $f(x)=1$ have no influence on the possibility $\mathring{f}$. So if the aim of an initial valuation f is to construct the related possibility function, it is no restrictive to assume that f is different from 1 in its domain. Indeed, $f(x)=1$ expresses lack of information about the event x in that we do not know reasons to disbelieve in x.

<u>Example 3.3</u> The initial valuation f is defined in an event e only and $f(e)=\alpha$. In this case we denote by p^e_α the possibility generated by f and

$$p^e_\alpha(x) = \begin{cases} 0 & \text{if } x=0 \; ; \\ \alpha & \text{if } x\le e \; ; \\ 1 & \text{otherwise.} \end{cases}$$

We denote by p^e the possibility p^e_1. If n^e_α and n^e are defined as in Example 2.1, then

$$\sim p^e_\alpha = n^{-e}_{1-\alpha} \quad ; \quad \sim n^e_\alpha = p^{-e}_{1-\alpha} \quad ; \quad \sim p^e = n^{-e} \quad ; \quad \sim n^e = p^{-e} \; .$$

References

L.Biacino, G.Gerla, (1984), Closure systems and L-subalgebras, *Information Sciences*, 32 , 181-195.

L.Biacino, G.Gerla, (1990), Generated Necessities, unpublished paper.

A.P.Dempster, (1967) , Upper and lower probabilities induced by a multivalued mapping, *Ann.Math. Statist.* 38, 325-339.

A.P.Dempster, (1968), A generalization of Bayesian inference, *J.Roy. Statist. Soc.* Ser.B 30, 205-247.

D.Dubois, H. Prade, (1988), Possibility Theory, Plenum Press, New York and London.

G. Shafer , (1976), A Mathematical Theory of Evidence, Princeton University Press, Princeton.

L.A.Zadeh, (1978) , Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, 1, 3-28.

Zhang Guangquan, (1988), Extension of possibility measure defined on an arbitrary nonempty class of L-fuzzy sets, *Busefall*, 37, 42-48.

---

* L.Biacino, Dip. Mat. ed Appl. "R.Caccioppoli", via Mezzocannone 8, 80134 Napoli, Italy.

** G.Gerla, Ist.Mat. Università della Basilicata, via N.Sauro 85, 85100 Potenza, Italy.