ON DECISION THEORY IN A FUZZY MEASURE ENVIRONMENT - SOME METHODOLOGICAL DIFFICULTIES

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1. Introduction

Remember that the standard situation in decision theory is the following: "Nature" chooses a certain state s from a universe S of states and the man chooses, as an answer, a certain action a from a universe A of actions. The so-called loss $m_L: S \times A \longrightarrow R^1$, i.e. $m_L(s,a)$, evaluates the action a for given s. In many real world situations observations $(x_1,\dots,x_n) =: \underline{x}$ are available, which give some information on the underlying state s to the experimenter and as actions suitable (decision-) functions $a(\underline{x})$ are used. In a probabilistic environment this is modelled by the assumption that the x_i follow a probability distribution which depends on the unknown state s, say $P_x(\underline{x}/s)$, and the decision function $a(\cdot)$ is chosen so that the risk

$$R_p(s,a) = \int m_L(s,a(\underline{x}))\, dP_x(\underline{x}/s) = \int m_L(s,t)\, dP_a(t/s) \qquad (1)$$

is minimized in some sense. Here $P_a(\cdot/s)$ is the probability measure of $a(\cdot)$ induced by $P_x(\cdot/s)$. In the following we will use only the relative loss or the degree of loss, i.e.

Assumption : $m_L: S \times A \longrightarrow [0,1]$

Then, m_L can be interpreted as the membership function of the fuzzy set L (called LOSS) and (1) appears as ZADEH's probability of the fuzzy event L, i.e.

$$R_p(s,a) = P_a(L/s) \qquad (2)$$

(see also NÄTHER /4/). Note that SUGENO /6/ has introduced a decision theory in a fuzzy measure environment. This essentially depends on the method used for extending the functional "fuzzy measure" to fuzzy sets. In the following we will sketch some different approaches and discuss the difficulties which arise for the evaluation of decisions.

2. Fuzzy measures for fuzzy sets

Remember that a fuzzy measure F on a universe X is defined by (see SUGENO /6/)

i) $F(\emptyset) = 0$, $F(X) = 1$

ii) $A \subset B \subset X \implies F(A) \leq F(B)$ (monotonicity)

iii) $A_i \subset A_{i+1}$ $(i=1,2,\ldots) \implies F(\bigcup A_i) = \lim_{i\to\infty} F(A_i)$ (continuity)

Now, the problem is how to define F for fuzzy sets A with the membership function m_A. Formally, F(A) is a certain expectation of m_A w.r.t. F. Following SUGENO /6/ we can use the SUGENO-integral, i.e.

$$F_{SU}(A) = \int_X m_A \circ F = \sup_{\alpha} \inf (\alpha, F(A_\alpha)) \tag{3}$$

where $A_\alpha = \{x \in X: m_A(x) \geq \alpha\}$ denotes the α-cut of A. Note that (3) can be applied to any fuzzy measure F.

Another possibility goes back to CHOQUET /2/ (see also WEBER /7/ and DUBOIS/PRADE /3/) and writes

$$F_{CH}(A) = \int_0^1 F(A_\alpha)\, d\alpha \, . \tag{4}$$

Note that (4) is not restricted to fuzzy measures, but can be used for general set functions provided that the integral exists.

SMETS /5/ only considers special fuzzy measures, the plausibility Pl and the associated belief Bel (connected by $Pl(B) = 1 - Bel(\bar{B})$) and defines for a fuzzy set A

$$Pl(A) = E^*(m_A) = \int_0^1 \alpha \, dBel(\bar{A}_\alpha) = - \int_0^1 \alpha \, dPl(A_\alpha)$$

$$Bel(A) = E_*(m_A) = \int_0^1 \alpha \, dPl(\bar{A}_\alpha) = - \int_0^1 \alpha \, dBel(A_\alpha)$$

which can be written in one formula by

$$F_{SM}(A) = - \int_0^1 \alpha \, dF(A_\alpha) \quad ; \; F = Pl \quad \text{or} \quad F = Bel \, . \tag{5}$$

Note that Smets and Choquet coincide if $F(A_{\alpha=1}) = 0$. This can be seen by use of the partial integration formula:

$$F_{CH}(A) = \int_0^1 F(A_\alpha)\, d\alpha = \alpha F(A_\alpha)\Big|_0^1 - \int_0^1 \alpha \, dF(A_\alpha) = F(A_{\alpha=1}) - \int_0^1 \alpha \, dF(A_\alpha)$$

Consider another special class of fuzzy measures, the so-called $\perp$-decomposable Archimedean measures (see WEBER /7/), characterized by a so-called additive generator $g:[0,1]\rightarrow[0,\infty)$. Such a fuzzy measure can be interpreted as a "distorted" probability measure P, i.e. $F = g^{-1}\circ P$, and for a fuzzy set A WEBER /7/ defines

$$F_{WE}(A) = g^{-1}\left(\int_X m_A \, dP \right) = g^{-1}(P(A)) \tag{6}$$

where P(A) denotes the probability of the fuzzy event A in the sense of ZADEH.

This (non-exhaustive) overview shows a multiplicity of definitions, all well-founded in their contexts. But, in general, they do not coincide. This appears as a difficulty in working with a decision theory in a fuzzy measure environment. Assume that the "observations" or pieces of information $(x_1,\ldots,x_n) = \underline{x}$ follow a fuzzy measure which depends on the state s of nature, say $F_{\underline{x}}(\underline{x}/s)$. Then, analogously to (2), the risk of a decision function $a(\cdot)$, principially, can be defined as

$$R_F(s,a) = F_a(L/s) \tag{7}$$

where $F_a(\cdot/s)$ is the fuzzy measure of $a(\cdot)$ induced by $F_{\underline{x}}(\cdot/s)$. But now the methodological difficulty consists in the non-unique definition of $F_a(L/s)$: What shall we prefer? The SUGENO-, CHOQUET-, SMETS- or WEBER-approach? The situation is complicated especially, if for certain fuzzy measures all four approaches can be applied, as discussed in the next section.

3. Some examples

Remember that YAGER's family $\{F_q\}_{q>0}$ of fuzzy measures is defined by (see YAGER /9/)

i) $F_q(\emptyset) = 0$, $F_q(X) = 1$

ii) $F_q(A\cup B) = \min\{[F_q(A)^{1/q} + F_q(B)^{1/q}]^q, 1\}$; $A\cap B = \emptyset$.

Note that F_q is $\perp$-decomposable and Archimedean with the generator $g_q(x) = x^{1/q}$. Thus, the "distorted" probability P,

$$F_q = g_q^{-1}\circ P = P^q \, , \tag{8}$$

is a special YAGER-fuzzy-measure. Note further that F_q from (8) is a belief function for $q > 1$ and a plausibility measure for $q < 1$ (see BERRES /1/ for integers q and WELLE /8/ for the general case).

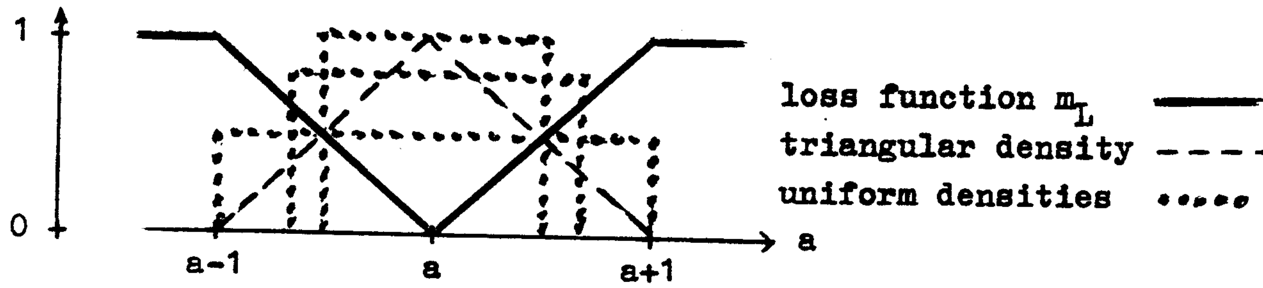
Consider a decision problem in an environment characterized by F_q from (8), i.e.

$$F_a(\cdot/s) = [P_a(\cdot/s)]^q \tag{9}$$

Since for $F_a(\cdot/s)$ all approaches from section 2 are applicable we obtain several risks (see also (7)):

$$R_F(s,a) = \begin{cases} \sup_\alpha \inf(\alpha, P_a(L_\alpha/s)^q) & \text{SUGENO} \\ \int_0^1 F_a(L_\alpha/s)\, d\alpha & \text{CHOQUET} \\ -\int_0^1 \alpha\, dF_a(L_\alpha/s) & \text{SMETS} \\ (P_a(L/s))^q & \text{WEBER} \end{cases} \tag{10}$$

For a numerical example consider, e.g., $q = 2$, a loss-membership-function $m_L(s,a) = \min(1, |s-a|)$ and several probability measures $P_a(\cdot/s)$ given by densities $p(a/s)$ characterizing several decision functions $a(\cdot)$. For illustration see the following figure and table.



density / p(a/s)= / risk	triangular max(0,1-|a-s|)	uniform $= \begin{cases} c & \text{for } -1/2c \leq a \leq 1/2c \\ 0 & \text{otherwise} \end{cases}$		
		c = 0,5	c = 0,8	c = 1,0
SUGENO	0,276	0,382	0,289	0,250
CHOQUET/SMETS	0,200	0,333	0,208	0,167
WEBER	0,111	0,250	0,098	0,063

Note that CHOQUET and SMETS coincide in the example since $F_a(L_{\alpha=1}/s) = 0$ (see the remark in connection with (5)).

Analysing the table, it is not so dangerous to have several absolute risk values. Difficulties arise, however, if the improvement of decisions should be estimated. Compare, e.g., the uniform decision with c=1 and the triangular: The SUGENO-risk shows a small, but the WEBER-risk a remarkable improvement. Moreover: Compare the triangular decision with the uniform c=0,8: WEBER prefers uniform, but all the other triangular. Thus, the several risks lead to several orderings between the decisions.

Note further that similar considerations are possible e.g. with SUGENO's λ-fuzzy-measures, for which all sketched approaches can be applied, too.

4. References

/1/ BERRES, M.: Glaubens- und Plausibilitätsgrade: Ein Beitrag zur Maß- und Integrationstheorie, Ph.D.Thesis, University of Mainz, 1984

/2/ CHOQUET, G.: Theory of capacities, Ann.Inst.Fourier, Univ. Grenoble, V(1954), 131-295

/3/ DUBOIS, D./ PRADE, H.: Evaluation of fuzzy sets and fuzzy integrations: A synthetic discussion, Abstracts of the 2nd Joint IFSA-EC-EURO-WG Workshop Vienna 1988, 131-135

/4/ NÄTHER, W.: Inference on parameters in possibility distributions, with special reference to possibilistic regression, BUSEFAL 37 (1988), 107-116

/5/ SMETS, PH.: The degree of belief in a fuzzy event, Inf.Sci. 25 (1981), 1-19

/6/ SUGENO, M.: Theory of fuzzy integrals and its application, Ph.D.Thesis, Tokyo Institute of Technology, 1974

/7/ WEBER, S.: $\perp$-decomposable measures and integrals for archimedean t-conorms $\perp$, J.Math.Anal.Appl.101 (1984), 114-138

/8/ WELLE, B.: Unscharfe Maße für unscharfe Mengen - Diskussion verschiedener Zugänge und Anwendungen in der Entscheidungstheorie, Diploma Thesis, Bergakademie Freiberg, Sektion Mathematik, 1989

/9/ YAGER, R.: On a general class of fuzzy connectives, Fuzzy Sets and Systems 4 (1980), 235-242