FIXED POINTS FOR SYMMETRIC FUZZY DECISIONS

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ABSTRACT

In the paper [1], $\varphi(\varphi')$ should be continuous function for symmetric fuzzy decisions. In this paper, we need not require the continuity of $\varphi(\varphi')$, and obtained some sufficient and necessary conditions that φ and φ' have the fixed point, and the relations among the fixed points of φ, φ', ψ and ψ' were studied.

KEYWORDS : Fuzzy Decision, Fixed Point.

I. SYMMETRIC FUZZY DECISIONS

Let U be an universe of discourse. Assume that we are given n fuzzy goals $M_1,\dots,M_n$ and m fuzzy constraints $C_1,\dots C_m$ in U. Then

$$D = (\cap_{i=1}^{n} M_i) \cap (\cap_{j=1}^{m} C_j) \quad (1)$$

is called symmetric fuzzy decision [2]. We want to solve the symmetric fuzzy decision problem, that is, to find $u^*\in U$ such that

$$D(u^*) = \sup_{u\in U} D(u) \quad (2)$$

Let $M=\cap_{i=1}^{n} M_i$ and $C=\cap_{j=1}^{m} C_j$, then

$$\sup_{u\in U} D(u) = \sup_{u\in U}[M(u) \wedge C(u)] \quad (3)$$

THEOREM 1.

$$\sup_{u\in U} D(u) = \sup_{\alpha\in[0,1]}[\alpha \wedge \sup_{u\in C_\alpha} M(u)] \quad (4)$$

$$\sup_{u\in U} D(u) = \sup_{\alpha\in[0,1]}[\alpha \wedge \sup_{u\in M_\alpha} C(u)] \quad (5)$$

where, C_α and M_α are α-level-sets of C and M respectively.

Remark. The Eq. (4) is from Theorem 6.3.1 of [1], and the Eq. (5) holds by the symmetry of M and C in the Eq. (3).

Let

$$\varphi(\alpha) = \sup_{u\in C_\alpha} M(u) \quad (6)$$

$$\psi(\alpha) = \alpha \wedge \varphi(\alpha) \quad (7)$$

$$\varphi'(\alpha) = \sup_{u\in M_\alpha} C(u) \quad (8)$$

$$\psi'(\alpha) = \alpha \wedge \varphi'(\alpha) \quad (9)$$

$$A = \{\, u\in U \mid M(u) \leqslant C(u) \,\} \quad (10)$$

$$B = \{\, u\in U \mid C(u) \leqslant M(u) \,\} \quad (11)$$

Similar to Theorem 6.3.2 of [1], we have

LEMMA 1. If a fixed point of $\varphi(\varphi')$ exists, then the fixed point is unique.

LEMMA 2. If $\bar{\alpha}$ is the fixed point of $\varphi(\varphi')$, then

(a). $\bar{\alpha} = 0$ iff $M = 0$ ($C = 0$).

(b). $A \supseteq C_{\bar{\alpha}}$ ($B \supseteq M_{\bar{\alpha}}$) $\neq \emptyset$.

Proof. We prove only the situation of φ, similarly, we can prove the situation of φ'.

(a). This is obvious.

(b). If $\bar{\alpha} = 0$, then $C_{\bar{\alpha}} = U \neq \emptyset$. If $\bar{\alpha} > 0$, then $C_{\bar{\alpha}} \neq \emptyset$. In fact, if $C_{\bar{\alpha}} = \emptyset$, then $\bar{\alpha} = \varphi(\bar{\alpha}) = \sup_{u \in C_{\bar{\alpha}}} M(u) = 0$.

By $\bar{\alpha} = \varphi(\bar{\alpha}) = \sup_{u \in C_{\bar{\alpha}}} M(u)$, we have that $\forall u \in C_{\bar{\alpha}}, C(u) \geq \bar{\alpha} \geq M(u)$. It follows that $C_{\bar{\alpha}} = A$. Q.E.D.

THEOREM 2. If $\bar{\alpha}$ is the fixed point of $\varphi(\varphi')$, then

$$\sup_{u \in U} D(u) = \psi(\bar{\alpha}) \ (\ \psi'(\bar{\alpha}) \ ) = \bar{\alpha} \tag{12}$$

Proof. By the Eq. (7), $\psi(\bar{\alpha}) = \bar{\alpha} \wedge \varphi(\bar{\alpha}) = \bar{\alpha} \wedge \bar{\alpha} = \bar{\alpha}$.

By the Eq. (4) and the Eq. (6), it suffices to show that

$$\sup_{\alpha \in [0,1]} [\alpha \wedge \varphi(\alpha)] = \psi(\bar{\alpha}) \tag{13}$$

If $\alpha \leq \bar{\alpha}$, then $\alpha \leq \bar{\alpha} = \varphi(\bar{\alpha}) \leq \varphi(\alpha)$. It follows that

$$\sup_{\alpha \leq \bar{\alpha}} [\alpha \wedge \varphi(\alpha)] = \sup_{\alpha \leq \bar{\alpha}} \alpha = \bar{\alpha} \tag{14}$$

If $\alpha > \bar{\alpha}$, then $\alpha > \bar{\alpha} = \varphi(\bar{\alpha}) \geq \varphi(\alpha)$. It follows that

$$\sup_{\alpha > \bar{\alpha}} [\alpha \wedge \varphi(\alpha)] = \sup_{\alpha > \bar{\alpha}} \varphi(\alpha) \leq \varphi(\bar{\alpha}) \tag{15}$$

By the Eq. (14) and the Eq. (15), we have

$$\begin{aligned} \psi(\bar{\alpha}) &= \bar{\alpha} \wedge \varphi(\bar{\alpha}) \leq \sup_{\alpha \in [0,1]} [\alpha \wedge \varphi(\alpha)] \\ &= \sup_{\alpha \leq \bar{\alpha}} [\alpha \wedge \varphi(\alpha)] \vee \sup_{\alpha > \bar{\alpha}} [\alpha \wedge \varphi(\alpha)] \\ &\leq \bar{\alpha} \vee \varphi(\bar{\alpha}) = \bar{\alpha} \vee \bar{\alpha} = \bar{\alpha} = \psi(\bar{\alpha}) \end{aligned}$$

It follows that the Eq. (13) holds.

Similarly, we can prove the situation of φ'. Q.E.D.

By Theorem 2, we transfer the symmetric fuzzy decision problem into the fixed point problem of φ or φ'.

II. FIXED POINT PROBLEMS OF φ AND φ'

THEOREM 3. $\varphi(\varphi')$ has the fixed point iff $\sup_{u \in A} M(u)$ $(\sup_{u \in B} C(u))$ is the unique fixed point of $\varphi(\varphi')$.

Proof. We prove only the situation of φ, similarly, we can prove the situation of φ'. Now, it suffices to show the necessity.

Let $\bar{\alpha}$ be the fixed point of φ. By Lemma 2, we have

$$\begin{aligned} \bar{\alpha} = \varphi(\bar{\alpha}) &= \sup_{u \in C_{\bar{\alpha}}} M(u) \leq \sup_{u \in A} M(u) = \sup_{u \in A - C_{\bar{\alpha}}} M(u) \vee \sup_{u \in C_{\bar{\alpha}}} M(u) \\ &= \sup_{u \in A - C_{\bar{\alpha}}} M(u) \vee \bar{\alpha} \leq \bar{\alpha} \vee \bar{\alpha} = \bar{\alpha} \end{aligned}$$

So $\bar{\alpha} = \sup_{u \in A} M(u)$, that is, $\sup_{u \in A} M(u)$ is the unique fixed point of φ by Lemma 1. Q.E.D.

THEOREM 4. If $\varphi(\varphi')$ has the fixed point, then

$$\sup_{u \in A} M(u) \geq (\leq) \sup_{u \in B} C(u)$$

Proof. We note the following Eq.

$$\sup_{u \in U} D(u) = \sup_{u \in A} M(u) \vee \sup_{u \in B} C(u) \tag{16}$$

and by Theorem 2 and Theorem 3, we have

$$\sup_{u \in U} D(u) = \sup_{u \in A} M(u) \ (\ \sup_{u \in B} C(u) \ )$$

So, $\sup_{u \in A} M(u) \geq (\leq) \sup_{u \in B} C(u)$. Q.E.D.

By Theorem 4, we have

COROLLARY 1. If φ and φ' have the fixed point, then

$$\sup_{u \in A} M(u) = \sup_{u \in B} C(u) = \sup_{u \in U} D(u)$$

THEOREM 5. If M (C) can not reach the supremum in A (B), then

$$\varphi(\bar{\alpha}) = \bar{\alpha} \ (\ \varphi'(\bar{\alpha}) = \bar{\alpha} \ ) \Longrightarrow \varphi'(\bar{\alpha}) = \bar{\alpha} \ (\ \varphi(\bar{\alpha}) = \bar{\alpha} \ )$$

Proof. We prove only the situation of φ. Similarly, we can prove the situation of φ'.

(a). $\varphi'(\bar{\alpha}) \geqslant \bar{\alpha}$.

In fact, $\varphi'(\bar{\alpha}) = \sup_{u\in M_{\bar{\alpha}}} C(u) = \sup_{u\in M_{\bar{\alpha}}\cap A} C(u) \vee \sup_{u\in M_{\bar{\alpha}}\cap A^c} C(u)$. By $u\in M_{\bar{\alpha}}\cap A$, it follows that $\bar{\alpha} \leqslant M(u) \leqslant C(u)$, so $\sup_{u\in M_{\bar{\alpha}}\cap A} C(u) \geqslant \bar{\alpha}$, and $\varphi'(\bar{\alpha}) \geqslant \bar{\alpha}$.

(b). $\varphi'(\bar{\alpha}) \leqslant \bar{\alpha}$.

In fact, if $\varphi'(\bar{\alpha}) > \bar{\alpha}$, then $\exists u_0 \in M_{\bar{\alpha}}$, $C(u_0) > \bar{\alpha}$. So $\bar{\alpha} \leqslant M(u_0)$ and $C(u_0) > \bar{\alpha}$. By $D = M \cap C$ and $\sup_{u\in U} D(u) = \bar{\alpha}$, we have

$$\bar{\alpha} \leqslant M(u_0) \wedge C(u_0) = D(u_0) \leqslant \bar{\alpha}$$

It follows that $\bar{\alpha} = M(u_0)$. By Theorem 3, we have

$$\sup_{u\in A} M(u) = \bar{\alpha} = M(u_0)$$

and by Lemma 2(b) and $C(u_0) > \bar{\alpha}$, we have

$$u_0 \in C_{\bar{\alpha}} \subseteq A$$

So, M can reach the supremum in A. This contradiction now shows that $\varphi'(\bar{\alpha}) \leqslant \bar{\alpha}$.

By (a) and (b), we have that $\varphi'(\bar{\alpha}) = \bar{\alpha}$. Q.E.D.

By Theorem 5, we have the following results

COROLLARY 2. If $\varphi(\bar{\alpha}) = \bar{\alpha}$ ($\varphi'(\bar{\alpha}) = \bar{\alpha}$), then

$$\psi'(\bar{\alpha}) = \bar{\alpha} \quad (\psi(\bar{\alpha}) = \bar{\alpha})$$

COROLLARY 3. If M can't reach the supremum in A and C in B, then

$$\varphi(\bar{\alpha}) = \bar{\alpha} \text{ iff } \varphi'(\bar{\alpha}) = \bar{\alpha}$$

COROLLARY 4. If M can't reach the supremum in A and C in B, then

$$\varphi(\varphi')(\bar{\alpha}) = \bar{\alpha} \Longrightarrow \sup_{u\in A} M(u) = \sup_{u\in B} C(u) = \sup_{u\in U} D(u) = \bar{\alpha}.$$

Proof. It can be proved immediately from Corollary 3, Corollary 1 and Theorem 2. Q.E.D.

THEOREM 6. Let $\sup_{u\in A} M(u)$ ($\sup_{u\in B} C(u)$) $= \bar{\alpha}$, then $\varphi(\varphi')$ has the fixed point iff $A (B) \supseteq C_{\bar{\alpha}} (M_{\bar{\alpha}})$ and $\psi(\bar{\alpha})$ ($\psi'(\bar{\alpha})$) $= \bar{\alpha}$.

Proof. We prove only the situation of φ. Similarly we can prove the situation of φ'.

Necessity follows from Lemma 2, Theorem 3 and Theorem 2. Sufficiency follows from the following

$$\varphi(\bar{\alpha}) = \sup_{u\in C_{\bar{\alpha}}} M(u) \leqslant \sup_{u\in A} M(u) = \bar{\alpha}$$

$$\psi(\bar{\alpha}) = \bar{\alpha} \wedge \varphi(\bar{\alpha}) = \bar{\alpha} \Longrightarrow \varphi(\bar{\alpha}) \geqslant \bar{\alpha}.$$

Q.E.D.

COROLLARY 5. Let $\sup_{u\in U} D(u) = \bar{\alpha}$. If $A \supseteq C_{\bar{\alpha}}$ and $B \supseteq M_{\bar{\alpha}}$, then the following equivalent:

(a). $\varphi(\bar{\alpha}) = \bar{\alpha}$

(b). $\psi(\bar{\alpha}) = \bar{\alpha}$

(c). $\varphi'(\bar{\alpha}) = \bar{\alpha}$

(d). $\psi'(\bar{\alpha}) = \bar{\alpha}$

Proof. By Theorem 6 (a) $\Longleftrightarrow$ (b) and (c) $\Longleftrightarrow$ (d). By Corollary (2), (a) $\Longrightarrow$ (d) and (c) $\Longrightarrow$ (b). Q.E.D.

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