CONNECTEDNESS OF GREY TOPOLOGICAL SPACE

(CONNECTEDNESS OF COMPOSITION FUZZY T.S.)

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ABSTRACT: In this paper the connectedness of grey topological spase is studied. The definitions and their theorems of connectedness of grey topological space are introduced.

KEYWORDS: Grey homeomorphic mapping, connected grey set and connected component.

I. INTRODUCTION

We introduced the definitions of grey topological space and of grey continuous mapping and preliminary studied compactness of grey topological space in [2]. We shall study the connectedness of grey topological space on this basis.

Definition 1: Let f be a bijective from the grey topological space $(X,\mathcal{J})$ to the grey topological space $(Y,\mathcal{J}')$. If f and f are grey continuous all, then f is called the grey homeomorphic mapping. $(X,\mathcal{J})$ and $(Y,\mathcal{J}')$ are called grey homeomorphic topological spaces, usually written $(X,\mathcal{J}) \cong (Y,\mathcal{J}')$.

Theorem 1: Let f be a mapping from the grey topological space $(X,\mathcal{J})$ to the grey topological space $(Y,\mathcal{J}')$. A is a grey subset of X and B, B_1 and B_2 are grey subsets of Y, then:

(1) $f^{-1}[B_1 \cup B_2] = f^{-1}[B_1] \cup f^{-1}[B_2]$.

(2) $f^{-1}[B_1 \cap B_2] = f^{-1}[B_1] \cap f^{-1}[B_2]$.

(3) $\overline{\mu}_A(x) \leqslant \overline{\mu}_{f^{-1}[f[A]]}(x)$, $\underline{\mu}_A(x) \leqslant \underline{\mu}_{f^{-1}[f[A]]}(x)$, $\forall x \in X$.

(4) $\overline{\mu}_{f[f^{-1}[B]]}(y) \leqslant \overline{\mu}_B(y)$, $\underline{\mu}_{f[f^{-1}[B]]}(y) \leqslant \underline{\mu}_B(y)$, $\forall y \in Y$.

(5) And f is the grey continuous mapping. If B is the closed subset of Y, then $f^{-1}[B]$ is the closed subset of X.

(6) And f is the grey continuous mapping, then

$\overline{\mu}_{f[\bar{A}]}(x) \leqslant \overline{\mu}_{\overline{f[A]}}(x)$, $\underline{\mu}_{f[\bar{A}]}(x) \leqslant \underline{\mu}_{\overline{f[A]}}(x)$, $\forall x \in X$.

(7) And f is the grey continuous mapping, then

$\bar{\mu}_{\overline{f^{-1}[B]}}(y) \le \bar{\mu}_{f^{-1}[\bar{B}]}(y)$, $\underline{\mu}_{\overline{f^{-1}[B]}}(y) \le \underline{\mu}_{f^{-1}[\bar{B}]}(y)$, $\forall y \in Y$.

Proof: We prove (5), (6) and (7) only.

(5) From reference [2] we have $f^{-1}[B^C] = [f^{-1}[B]]^C$.

Hence if B is the grey closed subset of Y, then B is the grey open subset of Y. So $f^{-1}[B^C] = [f^{-1}[B]]^C$ is the grey open subset of X, hence $f^{-1}[B]$ is the grey closed subset of X.

(6) From (3) we have $\bar{\mu}_A(x) \le \bar{\mu}_{f^{-1}[f[A]]}(x)$, $\forall x \in X$. And it is obvious $\bar{\mu}_{f^{-1}[f[A]]}(x) \le \bar{\mu}_{f^{-1}[\overline{f[A]}]}(x)$, $\forall x \in X$, hence $\bar{\mu}_A(x) \le \bar{\mu}_{f^{-1}[f[A]]}(x) \le \bar{\mu}_{f^{-1}[\overline{f[A]}]}(x)$, $\forall x \in X$. Also from (5) we have $f^{-1}[\overline{f[A]}]$ being the grey closed set, hence $\bar{\mu}_{\bar{A}}(x) \le \bar{\mu}_{f^{-1}[\overline{f[A]}]}(x)$, $\forall x \in X$. So $\bar{\mu}_{f[\bar{A}]}(x) \le \bar{\mu}_{\overline{f[A]}}(x)$ $\forall x \in X$.

In the same way we can also prove $\underline{\mu}_{f[\bar{A}]}(x) \le \underline{\mu}_{\overline{f[A]}}(x)$ $\forall x \in X$.

(7) From (6) we have $\bar{\mu}_{f[\overline{f^{-1}[B]}]}(y) \le \bar{\mu}_{\overline{f[f^{-1}[B]]}}(y) \le \bar{\mu}_{\bar{B}}(y)$, $\forall y \in Y$, hence $\bar{\mu}_{\overline{f^{-1}[B]}}(y) \le \bar{\mu}_{f^{-1}[\bar{B}]}(y)$, $\forall y \in Y$.

In same way we can also prove $\underline{\mu}_{\overline{f^{-1}[B]}}(y) \le \underline{\mu}_{f^{-1}[\bar{B}]}(y)$, $\forall y \in Y$.

Theorem 2: Let $(X, \Im)$ be a grey topological space. The grey subset A of X is the closed set if and only if $A = \bar{A}$.

Theorem 3: Let X be a whole grey set, a be a grey point of X, then $X = \bigcup_{a \in X} \{a\}$.

Definition 2: Let $(X, \Im)$ be a grey topological space, A and B be grey subsets of X. If $\bar{A} \cap B = A \cap \bar{B} = \emptyset$, then A and B are called separated.

II. CONNECTEDNESS OF GREY TOPOLOGICAL SPACE

Definition 3: Let $(X, \Im)$ be a grey topological space, C be a grey subset of X. If there do not exists non—vacuous separated grey subsets A and B of X such that $C = A \cup B$, then C is called the connected grey set.

If the whole grey set X is the connected set, then $(X, \Im)$ is called the connected grey topological space.

Theorem 4: Let $(X, \Im)$ be a grey topological space, the following are equivalent:

(1) $(X, \Im)$ is not connected.

(2) There exists non-vacuous grey closed sets A and B such that $A \cup B = X$ and $A \cap B = \emptyset$.

(3) There exists non-vacuous grey open sets A and B such that $A \cup B = X$ and $A \cap B = \emptyset$.

Proof: If $(X,\mathfrak{J})$ is not connected, then there exists grey subsets A and B of X such that $A \cup B = X$ and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

$\rightarrow$ $A \cup B = X$ and $A \cap B = \emptyset$. Where

$\bar{A} = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cap B) = (\bar{A} \cap A) \cup \emptyset = \bar{A} \cap A = A.$

$\bar{B} = \bar{B} \cap (\bar{B} \cup A) = (\bar{B} \cap B) \cup (\bar{B} \cap A) = (\bar{B} \cap B) \cup \emptyset = \bar{B} \cap B = B.$

Hence A and B are grey closed sets all. So (1)$\rightarrow$(2).

It is obvious (2)$\rightarrow$(1). Hence (1)$\longleftrightarrow$(2).

If (2) is correct, then there exists non-vacuous grey closed sets C and D such that $C \cup D = X$ and $C \cap D = \emptyset$.

$\rightarrow$ $\max\{\underline{\mu}_C(x), \underline{\mu}_D(x)\} = 1$ and

$\min\{\bar{\mu}_C(x), \bar{\mu}_D(x)\} = 0$. $\forall x \in X$.

Also $\bar{\mu}_C(x) = 1 - \underline{\mu}_{C^c}(x)$, $\underline{\mu}_C(x) = 1 - \bar{\mu}_{C^c}(x)$,

$\bar{\mu}_D(x) = 1 - \underline{\mu}_{D^c}(x)$, $\underline{\mu}_D(x) = 1 - \bar{\mu}_{D^c}(x)$, $\forall x \in X$.

Then $\max\{1 - \bar{\mu}_{C^c}(x), 1 - \bar{\mu}_{D^c}(x)\} = 1$ and

$\min\{1 - \underline{\mu}_{C^c}(x), 1 - \underline{\mu}_{D^c}(x)\} = 0, \forall x \in X$.

$\rightarrow$ Of $\bar{\mu}_{C^c}(x)$ and $\bar{\mu}_{D^c}(x)$ $(\forall x \in X)$ at least one must be equal to $0 \wedge$ of $\underline{\mu}_{C^c}(x)$ and $\underline{\mu}_{D^c}(x)$ $(\forall x \in X)$ at least one must be equal to 1.

$\rightarrow$ $\min\{\bar{\mu}_{C^c}(x), \bar{\mu}_{D^c}(x)\} = 0 \wedge \max\{\underline{\mu}_{C^c}(x), \underline{\mu}_{D^c}(x)\} = 1$.

$\rightarrow$ $\max\{\underline{\mu}_{C^c}(x), \underline{\mu}_{D^c}(x)\} = \max\{\bar{\mu}_{C^c}(x), \bar{\mu}_{D^c}(x)\} = 1$

and $\min\{\bar{\mu}_{C^c}(x), \bar{\mu}_{D^c}(x)\} = \min\{\underline{\mu}_{C^c}(x), \underline{\mu}_{D^c}(x)\} = 0$.

$\rightarrow$ $C^c \cup D^c = X$ and $C^c \cap D^c = \emptyset$.

Let $A = C^C$ and $B = D^C$. Since C^C and D^C are grey closed sets, hence A and B are grey closed sets.

So there exists non-vacuous grey open sets A and B such that $A \cup B = X$ and $A \cap B = \emptyset$. So (2)$\rightarrow$(3).

In the same way we can also have (3)$\rightarrow$(2). Hence (2)$\longleftrightarrow$(3).

Theorem 5: Let $(X,\mathfrak{J})$ be a grey topological space, a be a grey point of X, then a is the connected grey set.

Theorem 6: Let $(X,\mathfrak{I})$ be a grey topological space, A be a connected grey subset and B be a grey subset of X. If $A \subseteq B \subseteq \bar{A}$, then B is connected.

Proof: Let $B = C \cup D$ and $\bar{C} \cap D = C \cap \bar{D} = \emptyset$. If we want to prove B is connected only need to prove $C = \emptyset$ or $D = \emptyset$.

Let $C_1 = A \cap C$ and $D_1 = A \cap D$, then $\bar{C}_1 \cap D_1 = C_1 \cap \bar{D}_1 = \emptyset$, and $C_1 \cup D_1 = A$ (actually $C_1 \cup D_1 = (A \cap C) \cup (A \cap D)$ $= [(A \cap C) \cup A] \cap [(A \cap C) \cup D] = A \cap [(A \cup D) \cap (C \cup D)]$ $= A \cap [(A \cup D) \cap B] = A)$.

Since A is connected, then $C_1 = \emptyset$ or $D_1 = \emptyset$.

If $C_1 = \emptyset$, then $A = D_1 = A \cap D$. $\rightarrow \bar{\mu}_A(x) = \min\{\bar{\mu}_A(x), \bar{\mu}_D(x)\}, \forall x \in X$. $\rightarrow \bar{\mu}_A(x) \leq \bar{\mu}_D(x), \forall x \in X$.

Also $B = C \cup D \subseteq \bar{A}$, hence $\bar{\mu}_{\bar{A}}(x) \geq \underline{\mu}_{\bar{A}}(x) \geq \bar{\mu}_B(x)$ $= \max\{\bar{\mu}_C(x), \bar{\mu}_D(x)\}, \forall x \in X$. And $\bar{\mu}_C(x)$ $= \min\{\bar{\mu}_C(x), \max\{\bar{\mu}_C(x), \bar{\mu}_D(x)\}\} = \min\{\bar{\mu}_C(x), \bar{\mu}_B(x)\}$ $\forall x \in X$, hence $\bar{\mu}_C(x) \leq \min\{\bar{\mu}_C(x), \bar{\mu}_{\bar{A}}(x)\}, \forall x \in X$.

So $\bar{\mu}_C(x) \leq \min\{\bar{\mu}_C(x), \bar{\mu}_{\bar{D}}(x)\}, \forall x \in X$.

Also $C \cap \bar{D} = \emptyset$, hence $\min\{\bar{\mu}_C(x), \bar{\mu}_{\bar{D}}(x)\} = 0$ $\forall x \in X$. $\rightarrow \bar{\mu}_C(x) = \underline{\mu}_C(x) = 0$ $(\forall x \in X)$. $\rightarrow C = \emptyset$.

If $D_1 = \emptyset$, In the same way we can also prove $D = \emptyset$.

So B is connected.

Theorem 7: Let $(X,\mathfrak{I})$ be a grey topological space, At $(t \in T)$ be grey subsets of X. If At $(t \in T)$ are connected and there exists $s \in T$ such that As and At are not separated for all $t \in T-\{s\}$, then $A = \bigcup_{t \in T} At$ is connected.

Corollary: Let $(x,\mathfrak{I})$ be a grey topological space, At $(t \in T)$ be connected grey subsets of X. If $\bigcap_{t \in T} At \neq 0$, then $\bigcup_{t \in T} At$ is connected.

Definition 4: Let $(X,\mathfrak{I})$ be a grey topological space, A be a maximal connected grey set (or if $A \subseteq B$ and B is connected, then A = B) . Then A is called the connected component of $(X,\mathfrak{J})$.

Theorem 8: Let $(x,\mathfrak{J})$ be a grey topological space, then:

(1) Union of all connerted components is equal to X.

(2) Different connected components are not joint.

Proof: (1) Choose any point $a\in X$, then a is the connected grey set. Let $\mathcal{B}=\{A|a\in A$ and A is the connected grey subset of X$\}$, then $\cap\mathcal{B}$ is non-vacuous. And let $\mathcal{U}=\cup\mathcal{B}$. From corollary of theorem 7 we have $\mathcal{U}$ is connected and $\mathcal{U}$ is the connected component.

Hence all there exists connected components of containing a, $\forall a\in X$. Also $\bigcup_{a\in X}\{a\}=X$, hence union of all connected components of $(X,\mathfrak{J})$ is equal to X.

(2) Let A and B are different connected components.

Suppose A and B are joint, then there exists $x\in X$ such that $\min\{\underline{\mu}_A(x),\underline{\mu}_B(x)\}>0$. $\rightarrow\min\{\bar{\mu}_A(x),\bar{\mu}_B(x)\}>0$. $\rightarrow A\cap B\neq\emptyset$. From corollary of theorem 7 we have $A\cup B$ is connected. In contradiction with A (or B) is the connected component. Hence A and B are not joint.

Theorem 9: Let f be a grey continuous mapping from the grey topological space $(X,\mathfrak{J})$ to the grey toplogical space $(Y,\mathfrak{J}')$. If A is a connected grey subset of $(X,\mathfrak{J})$, then $f[A]$ is the connected grey subset of $(Y,\mathfrak{J}')$.

Proof: If $f[A]=B\cup C$ and $\bar{B}\cap C=B\cap\bar{C}=\emptyset$.

And let $E=f^{-1}[B]$ and $F=f^{-1}[C]$, then $\bar{\mu}_A(x)\leqslant\bar{\mu}_{f^{-1}[f[A]]}(x)$ $=\bar{\mu}_{f^{-1}[B\cup C]}(x)=\bar{\mu}_{f^{-1}[B]\cup f^{-1}[C]}(x)=\bar{\mu}_{E\cup F}(x)=\max\{\bar{\mu}_E(x),\bar{\mu}_F(x)\}$, $\forall x\in X$. Also $\bar{\mu}_{\bar{E}}(x)=\bar{\mu}_{\overline{f^{-1}[B]}}(x)\leqslant\bar{\mu}_{f^{-1}[\bar{B}]}(x)$ and $\bar{\mu}_{\bar{F}}(x)=\bar{\mu}_{\overline{f^{-1}[C]}}(x)\leqslant\bar{\mu}_{f^{-1}[\bar{C}]}(x)$, $\forall x\in X$, hence $\bar{\mu}_{\bar{E}\cap F}(x)=$ $\min\{\bar{\mu}_{\bar{E}}(x),\bar{\mu}_F(x)\}\leqslant\min\{\bar{\mu}_{f^{-1}[\bar{B}]}(x),\bar{\mu}_{f^{-1}[C]}(x)\}=\bar{\mu}_{f^{-1}[\bar{B}]\cap f^{-1}[C]}(x)$ $=\bar{\mu}_{f^{-1}[\bar{B}\cap C]}(x)=\bar{\mu}_{f^{-1}[\emptyset]}(x)=\bar{\mu}_{\emptyset}(x)=0,\forall x\in X.$ $\rightarrow\bar{\mu}_{\bar{E}\cap F}(x)=\underline{\mu}_{\bar{E}\cap F}(x)=0,\forall x\in X.\rightarrow\bar{E}\cap F=\emptyset.$

In the same way we can also prove $E\cap\bar{F}=\emptyset$.

And let $G=A\cap E$ and $H=A\cap F$, hence $A=G\cup H$ and $\bar{G}\cap H$

$= G \cap \bar{H} = \emptyset$. Since A is connected, hence $G = \emptyset$ or $H = \emptyset$.

Suppose $G = \emptyset$ (without loss of generality), then

$$\bar{\mu}_A(x) = \bar{\mu}_H(x) = \bar{\mu}_{A\cap F}(x) = \min\{ \bar{\mu}_A(x), \bar{\mu}_F(x) \}$$
$$\leq \bar{\mu}_F(x), \forall x \in X. \rightarrow \bar{\mu}_{f[A]}(X) = \sup_{x\in f^{-1}(y)}\{\bar{\mu}_A(x)\} \leq \sup_{x\in f^{-1}(y)}\{\bar{\mu}_F(x)\}$$
$$= \bar{\mu}_{f[F]}(x) = \bar{\mu}_{f[f^{-1}[C]]}(x) \leq \bar{\mu}_C(x), \forall x \in X.$$

Also $f[A] = B \cup C$, hence $\bar{\mu}_{f[A]}(x) = \max\{\bar{\mu}_B(x), \bar{\mu}_C(x)\}$ $\forall x \in X$. $\rightarrow \bar{\mu}_B(x) = \min\{\bar{\mu}_B(x), \bar{\mu}_{f[A]}(x)\} \leq \min\{\bar{\mu}_B(x), \bar{\mu}_C(x)\} = \bar{\mu}_{B\cap C}(x), \forall x \in X$. From $\bar{B} \cap C = \emptyset$ we have $B \cap C = \emptyset$, then $\bar{\mu}_B(x) \leq \bar{\mu}_{B\cap C}(x) = 0$ $(\forall x \in X)$ $\rightarrow \bar{\mu}_B(X) = \underline{\mu}_B(X) = 0$ $(\forall x \in X)$. $\rightarrow B = \emptyset$. So $f[A]$ is connected.

Corollary : Connectedness of the grey topological space is the topological property of the space. Or if f is a grey homemorphic mapping from the grey topological space $(X, \mathfrak{J})$ to the grey topological space $(Y, \mathfrak{J}')$. Then $(X, \mathfrak{J})$ is connected if and only if $(Y, \mathfrak{J}')$ is connected.

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