A NOTE TO THE FUZZY OBSERVABLES

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Abstract. We define a fuzzy observable on a fuzzy measure space. The range of a fuzzy observable is contained in a soft fuzzy sub-σ- algebra on which the fuzzy measure is a fuzzy P-measure.

Let (X, S, P) be a probability space and let $f: X \longrightarrow R$ be a S-measurable random variable. Then its inverse $f^{-1}: \mathcal{B} \longrightarrow S$ ($\mathcal{B}$ is the family of all Borel subsets of real line) fulfille the next conditions:

(C1) $P(f^{-1}(R)) = 1$

(C2) f^{-1} is a σ-homomorphism, i.e.

i) $\forall E \in \mathcal{B}: f^{-1}(E^c) = (f^{-1}(E))'$, where $E^c = R-E$ and $A' = X-A$

ii) $\forall \{E_i\} \subset \mathcal{B} : f^{-1}(\bigcup E_i) = \bigcup f^{-1}(E_i)$

(C3) if $\{E_i\} \subset \mathcal{B}$ are pairwise disjoints, then

$$P(f^{-1}(\bigcup E_i)) = \sum P(f^{-1}(E_i)) \quad .$$

We propose to use a fuzzy version of the conditions (C1) - (C2) for a definition of a fuzzy random variable. A similar approach was used by Dvurečenskij and Riečan in [1] . In accordance with [1] we will call a fuzzy modification of f^{-1} as a fuzzy observable.

Let (X, σ) be a fuzzy measurable space in sense of Klement et all [2] , i.e. X is a nonempty set and σ is a fuzzy σ-algebre of fuzzy subsets of $X, \sigma \subset [0 , 1]^X$, and

1) $0_X \in \sigma$, 2) $\mu \in \sigma \Rightarrow \mu' \in \sigma$, 3) $\{\mu_i\} \subset \sigma \Rightarrow \bigvee \mu_i \in \sigma$

Throughout this paper we will use the classical Zadeh's fuzzy connectives (see [4]), i.e.

$\vee \mu_i = \sup \mu_i$, $\wedge \mu_i = \inf \mu_i$, $\mu' = 1 - \mu$.

Let m be a measure on σ, $m: \sigma \longrightarrow [0, 1]$, satisfying

$m(0_X) = 0$ and $m(1_X) = 1$.

Definition 1. A fuzzy observable x on fuzzy measure space (X, σ, m) is a σ-homomorphism $x: \mathcal{B} \longrightarrow \sigma$ fulfilling the next conditions:

(F1) $m(x(R)) = 1$

(F2) i) $\forall E \in \mathcal{B}: x(E^c) = (x(E))'$

ii) $\forall \{E_i\} \subset \mathcal{B}: x(\cup E_i) = \vee x(E_i)$

(F3) if $\{E_i\} \subset \mathcal{B}$ are pairwise disjoints, then

$m(x(\cup E_i)) = \sum m(x(E_i))$.

Note that the condition (C2) satisfied by a f^{-1} determines an unique f, so that the conditions (C1) and (C3) are superfluous. In the fuzzy case is the situation different in the next sense: if we omit (F1) or (F3), our definition of a fuzzy observable can lead to an absurd fuzzy observable z, for which for all Borel subset E we have $z(E) = (1/2)_X$. On the other hand, if we take a stronger condition (F1a): $x(R) = 1_X$, then x(E) are crisp subsets of X and x is an inverse of a random variable.

Proposition 1. Let x be a fuzzy observable on fuzzy measure space (X, σ, m) and let $M_x = \{0_X, 1_X\} \cup \{x(E), E \in \mathcal{B}\}$. Then M_x is a soft fuzzy sub-σ-algebra of σ .

Proof. A soft fuzzy σ-algebra (see e.g. [3]) is a fuzzy σ-algebra not containing the fuzzy subset $(1/2)_X$. So it is enough to prove that M_x is closed under complementation and under countable unions and $(1/2) \notin M_x$.

i) Let $\mu \in M_x$. If $\mu \in \{0_X, 1_X\}$, then evidently $\mu' \in M_x$.

If $\mu = x(E)$ for a Borel subset E, then $\mu' = x(E^c) \in M_x$.

ii) Let $\{\mu_i\} \subset M_X$. If $1_X \in \{\mu_i\}$, then $\bigvee \mu_i = 1_X \in M_X$. If $1_X \notin \{\mu_i\}$, then either all μ_i are 0_X and $\bigvee \mu_i = 0_X \in M_X$, or $\bigvee \mu_i = \bigvee_{\mu_i \neq 0_X} \mu_i = \bigvee_{\mu_i \neq 0_X} x(E_i) = x(E) \in M_X$, where $E = \bigcup_{\mu_i \neq 0_X} E_i$.

iii) If $(1/2)_X \in M_X$, then $(1/2)_X = x(E)$ for some $E \in \mathcal{B}$. But then $x(E^c) = (1/2)_X = (1/2)_X$, $x(E \cup E^c) = x(R) = (1/2)_X$, so that $m((1/2)_X) = 1$. On the other hand, $x(E \cap E^c) = x(\emptyset) = (1/2)_X$, so that for any natural k we obtain taking $E_1 = \ldots = E_k = \emptyset$:
$1 = m(x(\emptyset)) = m(x(\bigcup E_i)) = \sum m(x(E_i)) = k$.
It follows $(1/2)_X \notin M_X$.

<u>Definition 2</u> (see e.g. [3]). A fuzzy P-measure m defined on a soft fuzzy σ-algebra σ is a mapping, $m: \sigma \longrightarrow [0, 1]$, fulfilling two next conditions:

i) $\forall \mu \in \sigma$: $m(\mu \vee \mu') = 1$

ii) $\forall \{\mu_i\} \subset \sigma$ of pairwise W-disjoint fuzzy subsets, i.e. $\mu_i \leq \mu_j'$ whenever $i \neq j$:
$m(\bigvee \mu_i) = \sum m(\mu_i)$.

<u>Proposition 2</u>. Under conditions of Proposition 1, m is a fuzzy P-mesure on M_X.

<u>Proof</u>.

i) Let $\mu \in M_X$. If $\mu \in \{0_X, 1_X\}$, then $\mu \vee \mu' = 1_X$ and $m(\mu \vee \mu') = m(1_X) = 1$. Let $\mu = x(E)$ for some $E \in \mathcal{B}$. Then $m(\mu \vee \mu') = m(x(E) \vee x(E^c)) = m(x(E \cup E^c)) = m(x(R)) = 1$.

ii) Let $\mu, \eta \in M_X$, $\mu \leq \eta'$. It is enough to take into account the case $\mu = x(E)$, $\eta = x(F)$ for some $E, F \in \mathcal{B}$. We have

$\mu = \mu \wedge \eta' = x(E \cap F^c)$,

$\eta = \eta \wedge \mu' = x(F \cap E^c)$,

$E \cap F^c$ and $F \cap E^c$ are disjoint , so that

$$m(\mu \vee \eta) = m(x((E \cap F^c) \cup (F \cap E^c))) = m(x(E \cap F^c)) + m(x(F \cap E^c)) = m(\mu) + m(\eta).$$

For a finite or countable system $\{\mu_i\} \subset M_X$ of pairwise W-disjoint fuzzy subsets we use the same method.

Remark. Our results approve the Dvurečenskij and Riečan's definition of a fuzzy observable [1] , where in the underlaying fuzzy measure space (X, σ, m) σ is a soft fuzzy σ-algebra and m is a fuzzy P-measure on σ . This enable to obtain many nice results, which for a more general definition may drop (especially the convergence theorems).

REFERENCES

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