# FUZZY VARIANT OF THE SIMPLE ITERATION METHOD

**Marian Matłoka**

Department of Mathematics, Academy of Economics,

ul. Marchlewskiego 146/150, 60-967 Poznań, Poland

## 1. Introduction.

The present work is devoted to a study of behaviour of the linear fuzzy transformation. This paper consists of three sections. The second section is devoted in general to interval and fuzzy numbers. Kaufmann and Gupta, [1], noticed a closed connection between the fuzzy numbers and intervals. It appears that if one takes fuzzy numbers from a certain class, every cut of them is an interval. Kaufmann and Gupta call those intervals "intervals of confidence". This close connection largely facilitates performing operations on fuzzy numbers. This fact strengthens our belief that in the nearest future we will wait to see a series of practical applications of the fuzzy analysis and computational methods, which at present are fit for interval analysis, will be addpted for the needs of fuzzy analysis.

In the third section we define a linear fuzzy transformation. Such a transformation is described by the matrix and vectors which elements are fuzzy numbers. For this transformation the simplest variety of the iteration method, the method of simple iteration is considered.

## 2. Preliminaries.

By an interval we mean a closed bounded set of "real" numbers

$$[a,b] = \{ x : a \le x \le b \}.$$

If A is an interval, we will denote its endpoints by $\underline{A}$ and $\overline{A}$. Thus, $A = [\underline{A}, \overline{A}]$. We will not distinguish between the degenerate interval $[a,a]$, and the real number, $a$.

We call two intervals equal if their corresponding endpoints are equal.

We can extend the order relation, $\leq$ , on the real line to intervals as follows:

$$A \leq B \text{ if and only if } \underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B} .$$

We can treat intervals A and B as numbers, adding them as follows: $A + B = C$, where $\underline{C} = \underline{A} + \underline{B}$ and $\overline{A} + \overline{B} = \overline{C}$.

More briefly, the rule for intervals addition is:

$$[\underline{A},\overline{A}] + [\underline{B},\overline{B}] = [\underline{A}+\underline{B},\overline{A}+\overline{B}] .$$

For the product of two intervals, we define

$$A\cdot B = \{ a\cdot b : a\in A, b\in B \} .$$

It is not hard to see that $A\cdot B$ is again an interval, whose endpoints can be computed from

$$\underline{A\cdot B} = \min (\underline{A}\cdot\underline{B}, \overline{A}\cdot\overline{B}, \underline{A}\cdot\overline{B}, \overline{A}\cdot\underline{B})$$

$$\overline{A\cdot B} = \max (\underline{A}\cdot\underline{B}, \overline{A}\cdot\overline{B}, \underline{A}\cdot\overline{B}, \overline{A}\cdot\underline{B}) .$$

The absolute value of the interval A we will define in the following way:

$$|A| = \max (|\underline{A}| ; |\overline{A}|) .$$

We introduce a metric topology for the set of intervals as follows. We define $d(A,B) = \max (|\underline{A}-\underline{B}|, |\overline{A}-\overline{B}|)$. We call $d(A,B)$ the distance between A and B.

A convex and normal fuzzy subset in R is called a fuzzy number. Let $L(R)$ be a set of all fuzzy numbers in R such that

- for any $X\in L(R)$, X is upper semicontinuous,
- for any $X\in L(R)$, supp X is compact.

From the above properties it follows that if $X \in L(R)$ then for any $\alpha \in [0,1]$, $X^{\alpha}$ is a compact interval in R, where

$$X^{\alpha} = \begin{cases} t : X(t) \geq \alpha & \text{if } \alpha \in (0,1] \\ t : X(t) > 0 & \text{if } \alpha = 0 . \end{cases}$$

The set $X^{\alpha}$ is called $\alpha$ - cut of the fuzzy number X.

Now, define $D : L(R) \times L(R) \longrightarrow R_+ \cup \{0\}$ by the equation

$$D(X,Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha),$$

where $d(X^\alpha, Y^\alpha)$ denotes the distance between the intervals $X^\alpha$ and $Y^\alpha$. It is easy to see that $D$ is a metric in $L(R)$ and $(L(R), D)$ is a complete metric space, (see [3]).

We can extend the order relation, $\leq$, in $L(R)$ as follows:

$X \leq Y$ if and only if for any $\alpha \in [0,1]$, $X^\alpha \leq Y^\alpha$.

Let $X$ and $Y$ be two fuzzy numbers from $L(R)$. Then

- the addition of $X$ and $Y$, $X + Y$, is a fuzzy number such that for any $\alpha \in [0,1]$ $(X+Y)^\alpha = X^\alpha + Y^\alpha$,
- the multiplication of $X$ and $Y$ is a fuzzy number, $X \cdot Y$, such that for any $\alpha \in [0,1]$ $(X \cdot Y)^\alpha = X^\alpha \cdot Y^\alpha$, (see [1]).

Now, let us consider the Cartesian Product $\underbrace{L(R) \times \ldots \times L(R)}_{m \text{ times}} = L^m(R)$.

In the set $L^m(R)$ we may define the metric in the following way:

- for any $X = (X_1, \ldots, X_m)$, $Y = (Y_1, \ldots, Y_m) \in L^m(R)$

1) $D_1(X,Y) = \max_{1 \leq k \leq m} D(X_k, Y_k)$,

2) $D_2(X,Y) = \sum_{k=1}^{m} D(X_k, Y_k)$,

3) $D_3(X,Y) = \sqrt{\sum_{k=1}^{m} [D(X_k, Y_k)]^2}$.

## 3. Fuzzy variant of the simple iteration method.

From the classical analysis we well known that the following theorem is true:

Theorem 3.1. Let $\mathcal{X}$ be a complete metric space, $F$ be the operator in $\mathcal{X}$ self-mapping $\mathcal{X}$ and for any $x$ and $y$ from $\mathcal{X}$

$$\rho(F(x), F(y)) \leq \alpha \cdot \rho(x,y)$$

where $\alpha < 1$. Then there exists the point $x^* \in \mathcal{X}$ such that

$$x^* = F(x^*).$$

The mapping $F$ is called a compression.

Let us consider the following equation

$$Y = A \cdot X + B \qquad (3.1)$$

where

- $A$ is an $m \times m$ matrix of the elements $A_{ij} \in L(R)$,
- $B$ is an m-vector with the elements $B_i \in L(R)$,
- $X$ and $Y$ are m-vectors which elements are equal $X_j \in L(R)$ and $Y_i \in L(R)$ respectively.

The equation (3.1) we will call the linear fuzzy transformation.

Now, for each metric considered in Preliminaries we will give the conditions, under which the linear fuzzy transformation is a compression. First let us consider the metric $D_1$ in $L^m(R)$. Let $X$ and $\tilde{X}$ be from $L^m(R)$ and let $Y = A \cdot X + B$ and $\tilde{Y} = A \cdot \tilde{X} + B$. Then we have

$$D_1(Y,\tilde{Y}) = \max_{1 \le i \le m} D(Y_i,\tilde{Y}_i) = \max_{1 \le i \le m} \sup_{0 \le \alpha \le 1} d(Y_i^\alpha,\tilde{Y}_i^\alpha) =$$

$$= \max_{1 \le i \le m} \sup_{0 \le \alpha \le 1} \max\{ | \sum_j (\min(A_{ij}^\alpha \cdot X_j^\alpha) - \min(A_{ij}^\alpha \cdot \tilde{X}_j^\alpha))|,$$

$$| \sum_j (\max(A_{ij}^\alpha \cdot X_j^\alpha) - \max(A_{ij}^\alpha \cdot \tilde{X}_j^\alpha))| \} .$$

Using the inequalities

$$| \min(A_{ij}^\alpha \cdot X_j^\alpha) - \min(A_{ij}^\alpha \cdot \tilde{X}_j^\alpha)| \le |A_{ij}^\alpha| \cdot d(X_j^\alpha,\tilde{X}_j^\alpha)$$

and

$$| \max(A_{ij}^\alpha \cdot X_j^\alpha - \max(A_{ij}^\alpha \cdot \tilde{X}_j^\alpha) | \le |A_{ij}^\alpha| \cdot d(X_j^\alpha,\tilde{X}_j^\alpha)$$

we obtain the required inequalities

$$D_1(Y,\tilde{Y}) \le \max_{1 \le i \le m} \sup_{0 \le \alpha \le 1} \max\{ \sum_{j=1}^m |A_{ij}^\alpha| \cdot d(X_j^\alpha,\tilde{X}_j^\alpha) \} \le$$

$$\le \max_{1 \le i \le m} \sup_{0 \le \alpha \le 1} \sum_{j=1}^m |A_{ij}^\alpha| \cdot d(X_j^\alpha,\tilde{X}_j^\alpha) \le$$

$$\leq \max_{1\leq i\leq m} \sup_{0\leq\alpha\leq 1} \sum_{j=1}^{m} |A_{ij}^{\alpha}| \cdot \max_{1\leq j\leq m} d(X_j^{\alpha},\tilde{X}_j^{\alpha}) \leq$$

$$\leq \max_{1\leq i\leq m} \sum_{j=1}^{m} |A_{ij}^{0}| \cdot D_1(X,\tilde{X}) \ ;$$

From the above inequality it can be deduced: in the space $L^m(R)$ with the metric $D_1$ the linear fuzzy transformation $Y = A\cdot X + B$ is a compression if

$$\max_{1\leq i\leq m} \sum_{j=1}^{m} |A_{ij}^{0}| < 1 \qquad (3.2)$$

Now, let us consider the metric $D_2$ in $L^m(R)$. Then for any $X$ and $\tilde{X}$ from $L^m(R)$ we have:

$$D_2(Y,\tilde{Y}) = \sum_{i=1}^{m} D(Y_i,\tilde{Y}_i) = \sum_{i=1}^{m} \sup_{0\leq\alpha\leq 1} d(Y_i^{\alpha},\tilde{Y}_i^{\alpha}) =$$

$$= \sum_{i=1}^{m} \sup_{0\leq\alpha\leq 1} \max\left(|\underline{Y}_i^{\alpha} - \underline{\tilde{Y}}_i^{\alpha}|, |\overline{Y}_i^{\alpha} - \overline{\tilde{Y}}_i^{\alpha}|\right) \leq$$

$$\leq \sum_{i=1}^{m} \sup_{0\leq\alpha\leq 1} \sum_{j=1}^{m} |A_{ij}^{\alpha}| \cdot d(X_j^{\alpha},\tilde{X}_j^{\alpha}) \leq$$

$$\leq \sum_{i=1}^{m} \sup_{0\leq\alpha\leq 1} \sum_{j=1}^{m} |A_{ij}^{\alpha}| \sum_{j=1}^{m} d(X_j^{\alpha},\tilde{X}_j^{\alpha}) \leq$$

$$\leq \sum_{i=1}^{m} \sup_{0\leq\alpha\leq 1} \sum_{j=1}^{m} |A_{ij}^{\alpha}| \cdot \sup_{0\leq\alpha\leq 1} \sum_{j=1}^{m} d(X_j^{\alpha},\tilde{X}_j^{\alpha}) \leq$$

$$\leq \sum_{i=1}^{m} \sup_{0\leq\alpha\leq 1} \sum_{j=1}^{m} |A_{ij}^{\alpha}| \sum_{j=1}^{m} \sup_{0\leq\alpha\leq 1} d(X_j^{\alpha},\tilde{X}_j^{\alpha}) \leq$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{m} |A_{ij}^{\circ}| \cdot \sum_{j=1}^{m} D(X_j,\tilde{X}_j) =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} |A_{ij}^{\circ}| \cdot D_2(X,\tilde{X})$$

So, from the above inequalities it follows that in the space $L^m(R)$ with the metric $D_2$ the linear fuzzy transformation (3.1) is a compression if

$$\sum_{i=1}^{m} \sum_{j=1}^{m} |A_{ij}^{\circ}| < 1 \qquad (3.3)$$

In the same way we obtain the condition in the space $L^m(R)$ with the metric $D_3$.

$$D_3(Y,\tilde{Y}) = \sqrt{\sum_{i=1}^{m}[D(Y_i,\tilde{Y}_i)]^2} = \sqrt{\sum_{i=1}^{m}[\sup_{0\le\alpha\le 1} d(Y_i^\alpha,\tilde{Y}_i^\alpha)]^2} \le$$

$$\le \sqrt{\sum_{i=1}^{m}[\sup_{0\le\alpha\le 1} \sum_{j=1}^{m}|A_{ij}^\alpha|\cdot d(X_j^\alpha,\tilde{X}_j^\alpha)]^2} \le$$

$$\le \sqrt{\sum_{i=1}^{m}[\sup_{0\le\alpha\le 1} \sum_{j=1}^{m}|A_{ij}^\alpha|\cdot \sup_{0\le\alpha\le 1} d(X_j^\alpha,\tilde{X}_j^\alpha)]^2} \le$$

$$\le \sqrt{\sum_{i=1}^{m}[\sup_{0\le\alpha\le 1} \sum_{j=1}^{m}|A_{ij}^\alpha|\cdot D(X_j,\tilde{X}_j)]^2} \le$$

$$\le \sqrt{\sum_{i=1}^{m}[\sum_{j=1}^{m}|A_{ij}^0|\ D(X_j,\tilde{X}_j)]^2} \le$$

$$\le \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}|A_{ij}^0|^2}\cdot D_3(X,\tilde{X}).$$

So, in the space $L^m(R)$ with the metric $D_3$ the linear fuzzy transformation (3.1) is a compression if

$$\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}|A_{ij}^0|^2} < 1 \qquad (3.4)$$

Let us note, that each of the conditions (3.2) - (3.4) is sufficient for the linear fuzzy transformation (3.1) to be a compression.

In accordance with the Theorem 3.1 at $\alpha < 1$ for transforming (3.1) there exists a point $X^* \in L^m(R)$ such that

$$X^* = A\cdot X^* + B.$$

If one takes an arbitrary point $X^{(0)} \in L^m(R)$, then on the basis of the conclusions of Theorem 3.1, at $\alpha < 1$, one can construct the sequence

$$X^{(l+1)} = A\,X^{(l)} + B \qquad (3.5)$$

which converges to $X^*$ with respect to the metric $D_1$, $D_2$ and $D_3$ respectively. With it

$$D_1(X^*,X^{(l+1)}) \le \alpha_1\cdot D_1(X^*,X^{(l)})$$

$$D_2(X^*,X^{(l+1)}) \le \alpha_2\cdot D_2(X^*,X^{(l)})$$

and

$$D_3(X^*, X^{(l+1)}) \leq \alpha_3 \cdot D_3(X^*, X^{(l)}) \text{ .}$$

Whence it follows, that for each metric $D_1, D_2, D_3$ there exists a sequence of convex closed regions $S_k^i = S(X^*, D_i^k)$, $i=1,2,3$, containing $X^{(k+1)}$ and, as a centre $X^*$. Note that the distance between $X^*$ and $X^{(k+1)}$ does not surpass $D_i^k = \alpha_i \cdot D_i(X^*, X^{(k)})$. For the sequence $X^{(l)}$ the inequalities

$$D_i(X^{(l)}, X^{(l+1)}) \leq \alpha_i^l \cdot D_i(X^{(0)}, X^{(1)})$$

and

$$D_i^l \leq \frac{\alpha_i^l}{1 - \alpha_i} D_i(X^{(0)}, X(1)) \qquad (3.6)$$

take place. From (3.6) it follows that the iteration process (3.5) allows to obtain an approximate solution of the linear fuzzy transformation (3.1) with any degree of accuracy.

## References


[1] Kaufmann, A and Gupta, M , Introduction to fuzzy arithmetic. Theory and Applications, New York, 1985.

[2] Moore, R.E. , Methods and Applications of Interval Analysis, SIAM, Philadelphia, 1979.

[3] Puri, M.L and Ralescu, D.A, Differentials for fuzzy functions, J.Math.Anal. Appl. 91, 552-558, 1983.