THE LAW OF LARGE NUMBERS FOR FUZZY NUMBERS*

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We study the problem: if $\xi_1,\xi_2,\ldots$ are fuzzy numbers with modal values $M_1,M_2,\ldots$, then what is the strongest t-norm for which

$$\lim_{n\to\infty} \mathrm{Nes}\left[m_n-\varepsilon \leqq \frac{\xi_1+\ldots\xi_n}{n} \leqq m_n+\varepsilon\right] = 1, \text{ for any } \varepsilon>0 \qquad (1)$$

where $m_n = (M_1+\ldots+M_n)/n$, the arithmetic mean $(\xi_1+\ldots+\xi_n)/n$ is defined via sup- t-norm convulution and Nes denotes necessity. If the sequence of fuzzy numbers $\xi_1,\xi_2,\ldots$ with modal values $M_1,M_2,\ldots$ satisfies (1) then we say that it obeys the law of large numbers.
It is shown that if $\xi_1,\xi_2,\ldots$ are noninteractive (i.e. independent) fuzzy numbers of symmetric triangular form with common width $\alpha>0$ and the t-norm is weaker than H_0 (i.e. the Hamacher-operator with zero parameter) then the sequence $\xi_1,\xi_2,\ldots$ obeys the law of large numbers.
On the other hand we prove that if we use min-norm in (1) then the sequence of noninteractive fuzzy numbers of symmetric triangular form does not obey the law of large numbers.

Keywords: Possibility,Probability,Necessity

1.DEFINITIONS.

A fuzzy number ξ is a fuzzy set of the real line $\mathbb{R}$ with an

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unimodal, normalized (i.e. there exists unique $a\in\mathbb{R}$ such that $\xi(a)=1$) and upper-semicontinuous membership function.
Given a subset $D\subset\mathbb{R}$, the grade of possibility of the statement "D contains the value of ξ" is defined by

$$\mathrm{Pos}(\xi\,|\,D) = \sup_{x\in D} \xi(x) \tag{2}$$

The quantity $1-\mathrm{Pos}(\xi\,|\,\bar{D})$, where $\bar{D}$ is the complement of D, is denoted by $\mathrm{Nes}(\xi\,|\,D)$ and is interpreted as the grade of necessity of the statement "D contains the value of ξ". It satisfies dual property with respect to (2):

$$\mathrm{Nes}(\xi\,|\,D) = 1 - \mathrm{Pos}(\xi\,|\,\bar{D}).$$

If $D=[a,b]\subset\mathbb{R}$ then instead of $\mathrm{Nes}(\xi\,|\,[a,b])$ we shall write $\mathrm{Nes}(a\leqq\xi\leqq b)$ and if $D=\{x\}, x\in\mathbb{R}$ we write $\mathrm{Nes}(\xi=x)$.
A function $T:[0,1]\times[0,1] \to [0,1]$ is said to be triangular norm (t-norm for short) iff T is commutative, associative, non-decreasing and $T(x,1)=x$, $x\in[0,1]$.
The function $H_\gamma:[0,1]\times[0,1] \to [0,1]$,where $\gamma>0$, defined by

$$H_\gamma(u,v) = \frac{uv}{\gamma+(1-\gamma)(u+v-uv)}$$

is called Hamacher-norm with paprameter γ (H_γ-norm for short).

A symmetric triangular fuzzy number ξ denoted by (a,α) is defined as $\xi(t)=1-|a-t|/\alpha$ if $|a-\alpha| \leqq t \leqq a$ and $\xi(t) = 0$ otherwise, where $a\in\mathbb{R}$ is the modal value and $\alpha>0$ is the width of ξ.

If ξ and η are noninteractive fuzzy numbers and T a t-norm, then their T-sum is defined as

$$(\xi+\eta)(z) = \sup_{x+y=z} T(\xi(x),\eta(y)),$$

Let T_1,T_2 be t-norms. We say that T_1 is weaker than T_2 (and write $T_1 \leq T_2$) if $T_1(x,y) \leq T_2(x,y)$ for each $x,y \in [0,1]$.

2.Chebyshev's Form of the Law of Large Numbers.

We shall provide a fuzzy analogue of the following theorem.
Chebyshev's theorem. If $\xi_1, \xi_2, \ldots$ are a sequence of pairwise independent random variables having finite variances bounded by one and the same constant

$$D\xi_1 \leq C,\ D\xi_2 \leq C, \ldots, D\xi_n \leq C, \ldots$$

and

$$M = \lim_{n\to\infty} \frac{M_1+\ldots+M_n}{n}$$

exists, then for any positive constant ε,

$$\lim_{n\to\infty} \mathrm{Prob}\left[\left|\frac{\xi_1+\ldots+\xi_n}{n} - \frac{M_1+\ldots+M_n}{n}\right| < \varepsilon\right] = 1$$

where $M_n = M\xi_n$ and Prob denotes probability.

3.RESULTS.

In this section we shall prove that if the t-norm T is weaker than H_0 and $\xi_1, \xi_2, \ldots$ are a sequence of noninteractive symmetric triangular fuzzy numbers with the common width α, then it obeys the law of large numbers.

LEMMA 1. If ξ, η are fuzzy numbers and $\xi \subseteq \eta$ (i.e. $\xi(x) \leq \eta(x)$, for each $x \in \mathbb{R}$) then

$$\mathrm{Nes}(\xi = x) \geq \mathrm{Nes}(\eta = x), \text{ for each } x \in \mathbb{R}.$$

PROOF. From the definition of necessity we have

$$\mathrm{Nes}(\xi=x) = 1 - \mathrm{Pos}(\xi|\mathbb{R}\setminus\{x\}) = 1 - \sup_{t\neq x}\xi(t) \geqq 1 - \sup_{t\neq x}\eta(t)$$

$$= \mathrm{Nes}(\eta=x).$$

Which ends the proof.

LEMMA 2. Let T_1, T_2 be t-norms and let ξ_1, ξ_2 be noninteractive fuzzy numbers. If $T_1 \leqq T_2$ then

$$(\xi_1+\xi_2)_1 \subseteq (\xi_1+\xi_2)_2$$

where $(\xi_1+\xi_2)_i$ denotes the T_i-sum of fuzzy numbers $\xi, \eta; i=1,2$.

PROOF. From the definition of t-sum we have

$$(\xi_1+\xi_2)_1(z) = \sup_{x+y=z} T_1(\xi_1(x), \xi_2(y)) \leqq$$

$$= \sup_{x+y=z} T_2(\xi_1(x), \xi_2(y)) = (\xi_1+\xi_2)_2(z).$$

Which ends the proof.

LEMMA 3. Let T be a t-norm and let $\xi_i=(M_i,\alpha)$, $i\in\mathbb{N}$. Then we have

$$\mathrm{Pos}\left(\frac{\xi_1+\ldots\xi_n}{n} = \frac{M_1+\ldots M_n}{n}\right) = 1, n\in\mathbb{N}$$

PROOF. From the definition of min-sum it follows that

$$\left(\frac{\xi_1+\ldots\xi_n}{n}\right)\left(\frac{M_1+\ldots M_n}{n}\right) = 1, \ n\in\mathbb{N}.$$

Which ends the proof.

THEOREM 1. (Law of Large Numbers for Fuzzy Numbers) Let $T\leqq H_0$ and let $\xi_i=(M_i,\alpha)$, $i\in\mathbb{N}$ be noninteractive fuzzy numbers, then for any $\varepsilon>0$,

(i) $$\lim_{n\to\infty}\mathrm{Nes}\left[m_n-\varepsilon\leqq\frac{\xi_1+\ldots\xi_n}{n}\leqq m_n+\varepsilon\right] = 1$$

(ii) $\text{Nes}\left[\lim_{n\to\infty}\frac{\xi_1+...+\xi_n}{n}=M\right]=1$

where $m_n = \frac{M_1+...+M_n}{n}$.

Proof. (i) If $\varepsilon \geqq \alpha$ then we get (i) trivially. Let $0<\varepsilon<\alpha$. From lemma 1 and lemma 2 it follows that we need to prove (i) only for $T=H_0$. By using Lagrange's multipliers method and the decomposition principle of fuzzy numbers [1] in order to determine the memberhip function of $(\xi_1+...\xi_n)/n$ we have

$$\left[\frac{\xi_1+...\xi_n}{n}\right](z) = \frac{1 - \frac{|m_n-z|}{\alpha}}{1 + (n-1)\frac{|m_n-z|}{\alpha}} \quad \text{if } |m_n-z| \leqq \alpha \text{ and}$$

$$\left[\frac{\xi_1+...\xi_n}{n}\right](z) = 0 \text{ otherwise.}$$

Finally, since

$$\text{Nes}\left[m_n-\varepsilon \leqq \frac{\xi_1+...\xi_n}{n} \leqq m_n+\varepsilon\right]=1-\text{Pos}\left[\frac{\xi_1+...\xi_n}{n}\,|\,(-\infty,m_n-\varepsilon)\cup(m_n+\varepsilon,\infty)\right]$$

$$1 - \sup_{x\in[m_n-\varepsilon,m_n+\varepsilon]}\left[\frac{\xi_1+...\xi_n}{n}\right](x) = 1 - \frac{1-\frac{\varepsilon}{\alpha}}{1+(n-1)\frac{\varepsilon}{\alpha}},$$

therefore,

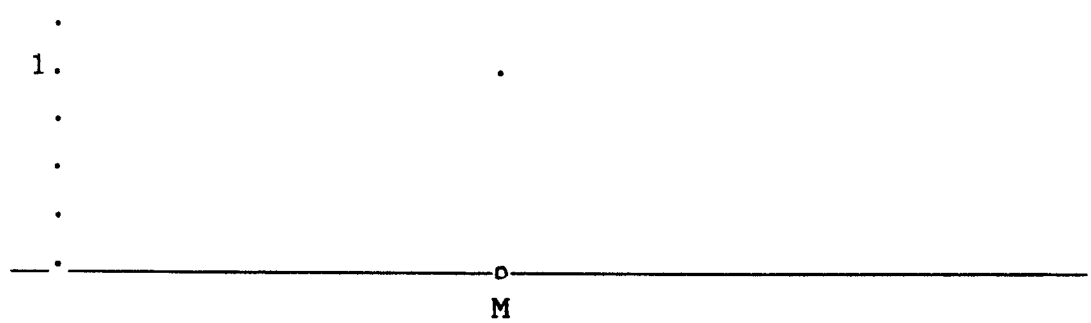
$$\lim_{n\to\infty} \operatorname{Nes}\left[m_n-\varepsilon \leqq \frac{\xi_1+\ldots\xi_n}{n} \leqq m_n+\varepsilon\right] = 1 - \lim_{n\to\infty} \frac{1-\frac{\varepsilon}{\alpha}}{1+(n-1)\frac{\varepsilon}{\alpha}} = 1.$$

(ii) By using the method of proof of (i) we get

$$\operatorname{Nes}\left[\lim_{n\to\infty}\frac{\xi_1+\ldots+\xi_n}{n} = M\right] = 1 - \sup_{t\neq M}(\lim_{n\to\infty}(\xi_1+\ldots\xi_n)/n)(t)$$

$$= 1 - \sup_{t\neq M} \chi_M(t) = 1,$$

where χ_M is the characteristic function of $\{M\}$.
Which ends the proof.



The limit distribution of $(\xi_1+\ldots+\xi_n)/n$ if $T\leqq H_0$.

REMARK 1. Theorem 1 can be interpreted as the law of large numbers for mutually T-related fuzzy variables [2]. Strong laws of large numbers for fuzzy random variables were proved in [4,5].
REMARK 2. Especially, if $T(u,v) = H_1(u,v)= uv$ then [3]

$$\lim_{n\to\infty}\operatorname{Nes}\left[m_n-\varepsilon \leqq \frac{\xi_1+\ldots\xi_n}{n} \leqq m_n+\varepsilon\right] = 1 - ((\xi_1+\ldots+\xi_n)/n)(m_n-\varepsilon)$$

$$= 1 - \lim_{n\to\infty} (1 - \varepsilon/\alpha)^n = 1.$$

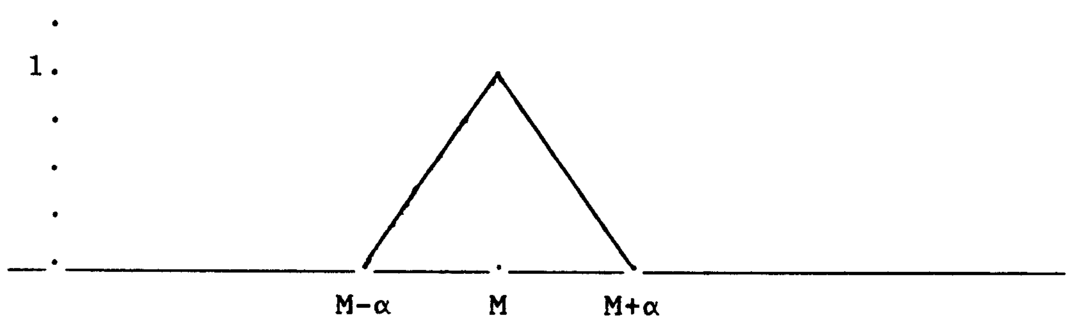
The following theorem shows that if T = "min" then the sequence of noninteractive fuzzy numbers $\xi_i=(M_i,\alpha)$, $i\in N$ does not obey the law of large numbers.

Theorem 2. Let $T(u,v)=\min\{u,v\}$ and $\xi_i=(M_i,\alpha), i\in N$. Then for any positive ε, such that $\varepsilon<\alpha$ we have

$$\lim_{n\to\infty} \mathrm{Nes}\left[m_n-\varepsilon \leq \frac{\xi_1+\ldots\xi_n}{n} \leq m_n+\varepsilon\right] = \frac{\varepsilon}{\alpha} .$$

$$\mathrm{Nes}\left[\lim_{n\to\infty}\frac{\xi_1+\ldots+\xi_n}{n} = M\right] = 0$$

PROOF. The proof of this theorem follows from the equality $\frac{1}{n}(\xi_1+\ldots\xi_n) = (m_n,\alpha)$.



The limit distribution of $(\xi_1+\ldots+\xi_n)/n$ if T ="min"

REMARK 3. Theorem 2 remains valid for symmetric fuzzy numbers $\xi_i=(M_i,\alpha)_{LL}$ $i\in N$, whenever the shape function L is continuous.

4. QUESTION.

Let T be a t-norm such that $H_0 < T <$ "min" and let $\xi_1, \xi_2, \ldots$ be a sequence of noninteractive symmetric triangular fuzzy numbers with common width $\alpha > 0$. Does this sequence obey the law of large numbers?

5. GUESS.

If $T \leq H_0$, then every sequences of compactely supporte noninteractive fuzzy numbers $\xi_1, \xi_2, \ldots$, such that the diameters (diam) of their supports are bounded by one and the same constant

$\mathrm{diam}(\mathrm{supp}\xi_1) \leq C$, $\mathrm{diam}(\mathrm{supp}\xi_2) \leq C, \ldots$ $\mathrm{diam}(\mathrm{supp}\xi_n) \leq C$,

obeys the law of large numbers for fuzzy numbers.

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