FUZZY MAPPINGS AND FUZZY PRODUCTION MODEL
Part 1: Fuzzy mappings

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1. Introduction.

The question of rational actions was formulated for the first time in political economy, but its scope by far exceeded the domain of economic studies. A sufficient condition to use the rule of rational actions is that the goal and means of actions be quatified, i.e. they should have the character of a "quantity" or at least "volume" so as to be measured or ordered. The rule of rational actions requires the possibility by determining the degree to which the goal has been attained in comparison with the starting level or previous state. A mode of utilizing means according with the rule of rational actions is called the optimal mode of utilization. However, to speak of the optimal mode of utilization of means and the extent to which a goal has been attained, first of all a given phenomenon should be characterized. Nowadays, economics makes use of a large number of powerful mathematical tools, to mention the papers of Morishima [5], Nikaido [6] , and Lancaster [3] as examples Moreover, certain specific features of described economic phenomena could not be taken into consideration for the simple reason that they evaded precise definitions, i.e. it was impossible to given their exact (numeric) values, and they could only be characterized with terms lite "a lot", "little" etc. Authors realized that descriptions of these phenomena were inadequate but the lack of a proper mathematical apparatus rendered precise considerations impossible. The turning point came in 1965 when Zadeh proposed the theory of fuzzy sets [13], which supplied a methodological apparatus more adequate than the classical one. For this reason it became extremely popular both among theoreticians and practician of various disciplines. The new ideas can be found in numerous papers published in all sorts of journals. For instance, the papers of Ponsard [7 - 12] present a fuzzy analogon of Debreu's analysis [2] and discuss the problem of production equilibrium with fuzzy constraint.

In this paper we present a fuzzy model of economic dynamics. The elaborated theory is a generalization of the theory of economic dynamics proposed by Makarov and Rubinov in [4], where the theory of multifunction is given the central position. Multifunctions were interpreted as certain technological transformations assigning a set of commodities to a set of production factors. It is assumed that the producer follows a precise behaviour pattern, by this we mean that the producer has complete information concerning the conditions of this productive activity and he has perfect command over both the set of inputs and the set of outputs. He realises the maximum profit allowed by the technoligical constraint which limits possible actions and by the given price system. In practic, the result of a production process is by nature imprecise. It follows that a technically possible production is more or less efficient. It is not advisable to partition the set of all possible productions into two classes: the efficient productions and the inefficient productions. As soon as at least one of the inputs does not have a maximal technical efficiency and / or as soon as at least one non-controlled input has an influence on the output quantities, the result of the productive combination is fuzzy. For a given combination the quantities of outputs obtained depend on the degree of efficiency of each of the controlled inputs and on the action of the factors which remain beyond the producer's control.

The above mentioned situation is difficult to describe but a fuzzy mapping seems to be a very useful tool in this respect. First we will present a definition of fuzzy mapping, some of its properties indispensable for further theory and economic interpretations of these properties. Next we will define a fuzzy economic model and we will consider a problem of optimality.

2. Some properties of fuzzy mappings.

Let X, Y, Z denote arbitrary but for further considerations fixed subsets. Next L(X), L(Y) and L(Z) denote respectively the families of all non-void fuzzy subset of X, Y and Z.

A fuzzy mapping, $f : X \to L(Y)$ say, is a mapping from X to L(Y), (see [1]).

In the other words, to each element $x \in X$ corresponds a fuzzy set f(x) from L(Y).

Instead of f(x) we will write f^x .

A set X will be identified with a set of commotity bundles which are the outlays at moment t. The fuzzy set f^x is a subset of the commodity space Y at moment τ $(t < \tau)$. A membership function f^x depicts the producer's subjective valuation of the authenticity of every point in Y obtained from x.

When $f^x(y) = 0$, the producer has no confidence at all with regard to the production state y which is obtained from x. If $f^x(y) = 0.3$, the producer has 30 percent confidence that he would operate at the state. Similarly, when $f^x(y) = 1$ the producer has full confidence to be at the production state.

For any fuzzy set $A \in L(X)$ we set

$$f^A(y) = \sup_{x \in X} (f^x(y) \wedge A(x)),$$

for any $y \in Y$.

Let us now consider three moments t, τ, θ $(t < \tau < \theta)$. A commodity bundle at moment t corresponds to a certain set of commodity bundles at moment θ. The character of the set, and the producer's valuation of authenticity of obtained commodity bundles depend on the technology. If temporal moment τ precedes θ then the assigment is not a direct one but depends on the technology operating in the interval from t to τ and on the technology operating in the interval from τ to θ. The below definition describes the way in which the quantities of commodities in the obtained bundles, and their producer's valuation of the authenticity depend on the quantities and producer's valuation of the authenticity of bundles in the moment τ.

A composition, $g \circ f : X \to L(Z)$ say, of two fuzzy mappings $f : X \to L(Y)$ and $g : Y \to L(Z)$ is a fuzzy mapping such that

$$(g \circ f)^x(z) = \sup_{y \in Y} (f^x(y) \wedge g^y(z)),$$

for any $x \in X$ and $z \in Z$.

Let us note, that for any $x \in X$ and any $z \in Z$

$$(g \circ f)^x(z) = g^{f^x}(z).$$

Let A and B denote two fuzzy subsets of some linear reference space X and α some real number. Owing the Zadeh's extension principle [14] , by $\alpha \cdot A$ and A+B such the fuzzy subsets are understand that for any $x \in X$

$$\alpha \cdot A(x) = \begin{cases} A(\frac{x}{\alpha}) & \text{for } \alpha \neq 0 , \\ 0 & \text{for } \alpha = 0 , \end{cases}$$

and

$$(A+B)(x) = \sup_{x'+x''=x} (A(x') \wedge B(x'')).$$

Let X, Y and Z denote the (crisp) convex cones.

A fuzzy mapping, $f : X \to L(Y)$ say, is called positively homogeneous iff $\forall\ x \in X$ and $\forall\ \alpha > 0$

$$f^{\alpha \cdot x} = \alpha \cdot f^x .$$

The above property means that proportional changes in quantities of the input and output commodity bundles do not influence the producer's valuation of their authenticity.

The next property of the fuzzy mapping to be discussed is its superadditivity. Let us consider two input commodity bundles x and u. For a non-fuzzy model it has beeb assumed that if the inputs x and u combined, a set of outputs is obtained, which is greater than the sum of sets of outputs bundles corresponding to particular inputs x and u. In our case, a similar assumption will be made to generate the producer's valuation of authenticity. This property is presented in the below definition.

A fuzzy mapping, $f : X \to L(Y)$ say, is called superadditive iff $\forall$ $x, u \in X$

$$f^{x+u} \supset f^{x} + f^{z} .$$

Corollary. If a fuzzy mapping $f : X \to L(Y)$ is superadditive and positively homogeneous then for any $x \in X$ the fuzzy set f^x is convex.

Theorem 2.1. If the fuzzy mappings $f : X \to L(Y)$ and $g : Y \to L(Z)$ are positively homogeneous then $g \circ f$ is positively homogeneous fuzzy mapping.

Proof. Let f and g denote two positively homogeneous fuzzy mappings. Then $\forall$ $x \in X$, $\forall$ $z \in Z$ and $\forall \alpha > 0$ there holds

$$(g \circ f)^{\alpha x}(z) = \sup_{y \in Y} (f^{\alpha x}(y) \wedge g^{y}(z)) =$$

$$= \sup_{\alpha y' \in Y} (f^{x}(\alpha y') \wedge g^{\alpha y'}(z)) = \sup_{y' \in Y} (f^{x}(y') \wedge g^{y'}(\frac{z}{\alpha})) =$$

$$= (g \circ f)^{x}(\frac{z}{\alpha}) = \alpha . (g \circ f)^{x}(z).$$

Theorem 2.2. If the fuzzy mappings $f: X \to L(Y)$ and $g: Y \to L(Z)$ are superadditive then $g \circ f$ is superadditive fuzzy mapping.

Proof. Let f and g denote two superadditive fuzzy mappings. Then $\forall\ x', x'' \in X$ and $\forall\ z \in Z$ we have

$$(g\circ f)^{x'+x''}(z) = \sup_{y\in Y} (f^{x'+x''}(y) \wedge g^{y}(z)) \geq$$

$$\geq \sup_{y\in Y} ((f^{x'}+f^{x''})(y) \wedge g^{y}(z)) =$$

$$= \sup_{y'+y''=y\in Y} ((f^{x'}+f^{x''})(y'+y'') \wedge g^{y'+y''}(z)) \geq$$

$$\geq \sup_{y'+y''=\in Y} ((f^{x'}(y') \wedge f^{x''}(y'') \wedge (g^{y'}+g^{y''})(z)) =$$

$$= \sup_{y'+y''=\in Y} ((f^{x'}(y') \wedge f^{x''}(y'') \wedge \sup_{z'+z''=z} (g^{y'}(z') \wedge g^{y''}(z''))$$

$$\geq \sup_{z'+z''=z} \sup_{y',y''\in Y} ((f^{x'}(y') \wedge g^{y'}(z') \wedge f^{x''}(y'') \wedge g^{y''}(z''))$$

$$= \sup_{z'+z''=z} (\sup_{y'\in Y} (f^{x'}(y') \wedge g^{y'}(z')) \wedge$$

$$\wedge \sup_{y''\in Y} (f^{x''}(y'') \wedge g^{y''}(z'')) =$$

$$= \sup_{z'+z''=z} ((g\circ f)^{x'}(z') \wedge (g\circ f)^{x''}(z'')) =$$

$$= ((g\circ f)^{x'} + (g\circ f)^{x''})(z).$$

By the graph of a fuzzy mapping $f\colon X \to L(Y)$ the fuzzy set is understood, G_f in symbol, of $X\times Y$ such that for any $x\in X$ and any $y\in Y$

$$G_f(x,y) = f^{x}(y).$$

Theorem 2.3. The graph of a positively homogeneous fuzzy mapping is a fuzzy cone i.e. $\forall\ (x,y)\in X\times Y$ and $\forall\ \alpha > 0$

$$G_f(x,y) = G_f(\alpha x, \alpha y).$$

In fact, taking into account the definition of the graph and the definition of a positively homogeneous fuzzy mapping we observe that for any $x\in X$, any $y\in Y$ and for every $\alpha > 0$ there holds

$$G_f(\alpha x,\alpha y) = f^{\alpha x}(\alpha y) = \alpha\cdot f^{x}(\alpha y) = f^{x}(y) = G_f(x,y).$$

Theorem 2.4. If a fuzzy mapping f: X $\to$ L(Y) is superadditive and positively homogeneous then its graph is a convex fuzzy set.

Proof. Let $x', x'' \in X$, $y', y'' \in Y$ and α, $\beta > 0$ such that $\alpha + \beta = 1$. Then there holds

$$G_f(\alpha x' + \beta x'', \alpha y' + \beta y'') = f^{\alpha x' + \beta x''}(\alpha y' + \beta y'') \geqslant$$

$$\geqslant (f^{\alpha x'} + f^{\beta x''})(\alpha y' + \beta y'') \geqslant f^{\alpha x'}(\alpha y') \wedge f^{\beta x''}(\beta y'') =$$

$$= \alpha f^{x'}(\alpha y') \wedge \beta f^{x''}(\beta y'') = f^{x'}(y') \wedge f^{x''}(y'') =$$

$$= G_f(x', y') \wedge G_f(x'', y'').$$

Now, let us assume that the reference space X,Y and Z are finite - dimensional Euclidean spaces and N denote the set of positive integers.

A fuzzy mapping, f: X $\to$ L(Y) say, is called closed iff its graph is a closed fuzzy set, i.e. the sendograph

$$\text{send } G_f = \{((x,y),r) : G_f(x,y) \geqslant r \,,\ x \in X, y \in Y,\ r \in (0,1\rangle\}$$

is closed (crisp) set.

Corollary. If f: X $\to$ L(Y) is closed fuzzy mapping, then for any $x \in X$, f^x is closed fuzzy set.

A fuzzy mapping f: X $\to$ L(Y) is sequentially bounded iff for any bounded sequence $S = \{x_n\}$ and any sequence $R = \{r_n\}$, $x_n \in X$, $r_n \in (0,1\rangle$, $n \in N$, the (crisp) set

$$f_{S,R} = \{\, y \in Y : f^{x_n}(y) \geqslant r_n,\ x_n \in S,\ r_n \in R,\ n \in N \,\}$$

is bounded.

Theorem 2.5. If f: X $\to$ L(Y) and g: Y $\to$ L(Z) are closed fuzzy mappings and f is sequentially bounded, then $g \circ f$ is a closed fuzzy mapping.

Proof. For any $n \in N$, $((x_n,z_n),r_n) \in \text{send } G_{g\circ f}$, and assume $((x_n,z_n),r_n) \to ((x_0,z_0),r_0)$ as $n \to \infty$, $x_n \in X$, $z_n \in Z$, $r_n \in (0,1\rangle$. We will prove that $((x_0,z_0),r_0) \in \text{send } G_{g\circ f}$. Because f is a sequentially bounded and closed fuzzy mapping, so for any $n \in N$ f^{x_n} is a closed and bounded (in Zadeh's sense [13]) fuzzy set. On the other hand, g is a closed fuzzy mapping, so $g_{\leftarrow}^{z_n}$ is a closed fuzzy set, where $g_{\leftarrow}$ denotes a converse fuzzy mapping to g i.e. for any $z \in Z$ and any $y \in Y$ $g^y(z) = g_{\leftarrow}^z(y)$. Therefore, for any r_n the (crisp) set

$$f_{z_n,r_n} = \{ y \in Y : f^{x_n}(y) \geqslant r_n \}$$

is a compact and the (crisp) set

$$g_{\leftarrow z_n,r_n} = \{ y \in Y : g_{\leftarrow}^{z_n}(y) \geqslant r_n \}$$

is closed. Because for any $n \in N$ $((x_n,z_n),r_n) \in \text{send } G_{g\circ f}$ we have

$$\sup_{y \in Y} (f^{x_n}(y) \wedge g^y(z_n)) \geqslant r_n \ .$$

Still, for $y \in Y \setminus (f_{x_n,r_n} \cap g_{\leftarrow z_n,r_n})$ there holds

$$f^{x_n}(y) \wedge g_{\leftarrow}^{z_n}(y) \leqslant r_n \ .$$

Therefore, for each n N we have

$$\sup_{y \in Y} (f^{x_n}(y) \wedge g^y(z_n)) = \sup_{y \in f_{x_n,r_n} \cap g_{\leftarrow z_n,r_n}} (f^{x_n}(y) \wedge g^y(z_n)).$$

Since for each $n \in N$ set $f_{x_n,r_n} \cap g_{\leftarrow z_n,r_n}$ is compact, there exists an element $\bar{y}_n$ such that

$$G_{g\circ f}(x_n,z_n) = G_f(x_n,\bar{y}_n) \wedge G_g(\bar{y}_n,z_n) \geqslant r_n \ .$$

Consequently, $G_f(x_n,\bar{y}_n) \geqslant r_n$ and $G_g(\bar{y}_n,z_n) \geqslant r_n$. Because f is a sequentially bounded fuzzy mapping, the sequence $\{\bar{y}_n\}_{n=1}^{\infty}$ is bounded too, and without losing generality we may assume that it converges to $\bar{y}$. But f and g are closed fuzzy mappings, so

$G_f(x_0,\bar{y}) \geq r_0$ and $G_g(\bar{y},z_0) \geq r_0$. But this means that $G_{g\circ f}(x_0,z_0) \geq r_0$, i.e. $((x_0,z_0),r_0) \in$ send $G_{g\circ f}$.

References

[1] Butnariu, D., Fixed points for fuzzy mappings, Fuzzy Sets and Systems 7, 191-207, 1982.

[2] Debreu, G., Theory and value, John Wiley, New York, 1959.

[3] Lancaster, K., Mathematical Economics, New York, 1968.

[4] Makarov, V.L. and Rubinov, A.M., Mathematical Theory of economic dynamic and equilibria, Moskva, 1973 (in Russian).

[5] Morishima, M., Equilibrium stability and growth, Oxford University Press, 1968.

[6] Nikaido, H., Convex structures and economic theory, Academic Press, 1968.

[7] Ponsard, C., An application of fuzzy subsets theory to the analysis of the consumer's spatial preferences, FSS 5, 235-244, 1981.

[8] Ponsard, C., On the imprecision of consumer's spatial preferences, Papers Regional Science Association 42, 59-71, 1979.

[9] Ponsard, C., Fuzzy economic spaces, Working Paper of the I.M.E.

[10] Ponsard, C., L'Equibre Spatial du Consommateur dans un Contexte Imprecis, Sistemi Urbani, 1-2, 107-133, 1981.

[11] Ponsard. C., Producer's spatial equilibrium with fuzzy constaint, European Journal of Operational Research 10, 302-313, 1982.

[12] Ponsard, C., Partial spatial equilibria with fuzzy constraints, Journal of Regional Science 22, 1982.

[13] Zadeh, L.A., Fuzzy sets, Information and Control 8, 338-353, 1965.

[14] Zadeh, L.A., The concept of a linguistic variable and its application to approximate reasoning. Part I, Information Sciences 8, 199-249.