ANTITONIC SET-VALUED MAPPINGS AND FUZZY SETS

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In this paper we have proved that antitonic set-valued mappings, from real interval to hypermeasurable space, are random sets, and have obtained that their falling shadom is fuzzy sets. We have used probability distribution function to express fuzzy sets, thus we can estimate the function of membership of fuzzy sets by means of empirical distribution functions.

Keywords: Hypermeasurable space, Antitonic set-valued mapping, Falling shadow, Set-valued statistics

1. Preparation

Let U be a set, for any $\mathbf{D} \subset P(U) = \{A \mid A \subset U\}$, write

$$P(\mathbf{D}) = \{ \mathbf{A} \mid \mathbf{A} \subset \mathbf{D} \}$$

Let $\mathbf{A}_n \in P(\mathbf{D})$ $(n \geq 1)$, $\mathbf{A} \in P(\mathbf{D})$, write

$$\bigcup_{n=1}^{\infty} \mathbf{A}_n = \{A \mid \exists A,\ A \in \mathbf{A}_n\}$$

$$\bigcap_{n=1}^{\infty} \mathbf{A}_n = \{A \mid \forall A,\ A \in \mathbf{A}_n\}$$

$$\mathbf{A}^c = \{A \mid A \notin \mathbf{A}\}$$

Definition 1-1 : Let $(U,\mathbf{D})$ be a measurable space, if $\widetilde{\mathbf{H}} \subset P(\mathbf{D})$ is a sigma-field with respect to the above operations, then $\widetilde{\mathbf{H}}$ is called sigma-hyperfield on the U. $(U,\mathbf{D},\widetilde{\mathbf{H}})$ is called hypermeasurable space.

It is apparent that $(U,\mathbf{D},P(\mathbf{D}))$ is hypermeasurable space.

Definition 1-2: Suppose that X and U are sets and let $\mathbf{D} \subset P(U)$, then mapping $f: X \to \mathbf{D}$ is called set-valued mapping from X to $\mathbf{D}$.

Definition 1-3: Let $(X,\mathbf{F})$ be a measurable space and $(U,\mathbf{D},\widetilde{\mathbf{H}})$ be a hypermeasurable space. If set-valued mapping $f: X - \mathbf{D}$ is $(\mathbf{F}\text{-}\widetilde{\mathbf{H}})$ measurable, i.e., for any $e \in \widetilde{\mathbf{H}}$

$$f^{-1}(e) = \{ x \mid x \in X, \quad f(x) \in e \} \in \mathbf{F}$$

then the f is called a random set on the U.

Definition 1-4: Suppose that $(X,\mathbf{F},P)$ is a probability space and $(U,\mathbf{D},\widetilde{\mathbf{H}})$ is a hypermeasurable space, let f be a random set, write

$$P_f(e) \triangleq P(f^{-1}(e)) \qquad (e \in \widetilde{\mathbf{H}})$$

then the P_f is called an induced distribution of the f on the $\widetilde{\mathbf{H}}$.

for any $u \in U$, write

$$\dot{u} = \{f(x) \mid x \in X, \ u \in f(x)\}$$

if $\dot{u} \in \widetilde{\mathbf{H}}$, then $P_f(\dot{u}) \triangleq P(f^{-1}(\dot{u}))$ is called a falling shadow of the f on this point u.

2. Falling Shadow of Antitonic Set-valued Mappings

In the following, I represents the real interval $[a,b]$, $(I,\mathbf{B})$ represents a Borel measurable space on the I, $(I,\mathbf{B},P)$ represents a probability space.

Definition 2-1: Let U be a set and $\mathbf{D} \subset P(U)$, if set-valued mapping $f: I \longrightarrow \mathbf{D}$ satisfies — for any $x_1, x_2 \in I$, if $x_1 \leqslant x_2$ $f(x_2) \subset f(x_1)$, then the f is called an antitonic set-valued mapping from I to $\mathbf{D}$.

For any $u \in U$, write

$$\dot{u} = \{ f(y) \mid y \in I, \quad u \in f(y) \}$$

Lemma 2-1: Let f be an antitonic set-valued mapping from I to $\mathbf{D}$, then any $u \in U$, there is $x = \sup\{ y \mid y \in I, \ u \in f(y)\}$, such that

$$f^{-1}(\dot{u}) = \begin{cases} [a,x) & \text{if } u \notin f(x) \\ [a,x] & \text{if } u \in f(x) \end{cases}$$

here $f^{-1}(\dot{u}) = \{ y \mid y \in I, \quad u \in f(y) \}$

Proof: If $\dot{u} = \phi$, i.e., for any $y \in I$ $u \notin f(y)$, then $f^{-1}(\dot{u}) = \phi$. We accept it as $x = a$, so $f^{-1}(\dot{u}) = \phi = [a,x)$

If $\dot{u} \neq \phi$, i.e., there is $y \in I$, such that $u \in f(y)$, write

$$x = \sup\{ y \mid y \in I, \quad u \in f(y) \}$$
$$= \sup\{ y \mid y \in I, \quad y \in f^{-1}(\dot{u}) \}$$

thus, for any $y \in f^{-1}(\dot{u})$, there is $y \leqslant x \Rightarrow f^{-1}(\dot{u}) \subset [a,x]$, for any $y < x$, there is $y \in f^{-1}(\dot{u}) \Rightarrow [a,x) \subset f^{-1}(\dot{u})$

Moreover, if $u \notin f(x)$, i.e., $x \notin f^{-1}(\dot{u}) \Rightarrow$ for any $y \in f^{-1}(\dot{u})$, there is $y < x \Rightarrow f^{-1}(\dot{u}) \subset [a,x)$

so, if $u \notin f(x)$, then $f^{-1}(\dot{u}) = [a,x)$

If $u \in f(x)$, i.e., $x \in f^{-1}(\dot{u}) \Rightarrow$ for any $y \leqslant x$, there is $y \in f^{-1}(\dot{u}) \Rightarrow$ $[a,x] \subset f^{-1}(\dot{u})$

so, if $u \in f(x)$, then $f^{-1}(\dot{u}) = [a,x]$.

Suppose that $(U,\mathbf{D})$ is a measurable space, let $f: I \to \mathbf{D}$ be a set-valued mapping, write

$$f(I) = \mathbf{A} = \{ f(x) \mid x \in I\} \subset \mathbf{D}$$
$$\dot{u} = \{A \mid u \in A \in \mathbf{A}\}$$
$$\hat{\mathbf{A}} = \{\dot{u} \mid u \in U\}$$
$$\sigma(\hat{\mathbf{A}}) = \cap \{\tilde{\mathbf{H}} \mid \hat{\mathbf{A}} \subset \tilde{\mathbf{H}},\ \tilde{\mathbf{H}} \text{ is a sigma-hyperfield on } U\}$$

then $\sigma(\hat{\mathbf{A}})$ is called sigma-hyperfield produced by the $\hat{\mathbf{A}}$; $(U,\mathbf{D},\sigma(\hat{\mathbf{A}}))$ is called hypermeasurable space produced by the $\hat{\mathbf{A}}$.

Theorem 2-1: Suppose that $(U,\mathbf{D})$ is a measurable space, let f be an antitonic set-valued mapping from I to $\mathbf{D}$, then the f is a random set on the U.

Proof: From the above lemma 2-1. we have

$$f^{-1}(\sigma(\hat{\mathbf{A}})) = \sigma(f^{-1}(\hat{\mathbf{A}})) = \sigma(\{ f^{-1}(\dot{u}) \mid u \in U\}) \subset \mathbf{B}.$$

Therefore the f is a random set on the U.

Let $(I,\mathbf{B},P)$ be a probability space, write

$$\bar{A}(u) \triangleq P_f(\dot{u}) = P(f^{-1}(\dot{u})) \quad (u \in U)$$

then falling shadow of the f is a fuzzy set on the U.

Example 2-1: We give a measurable space $(U,P(U))$, let $B \subset A \subset U$ and $x_0 \in I$.

Suppose mapping $f: I \to P(U)$ is that if $x < x_0$, then $f(x) = A$; if $x \geqslant x_0$, then $f(x) = B$,

so, for any $u \in B$, $f^{-1}(\dot{u}) = I$, then $P_f(\dot{u}) = P(I) = 1$

for any $u \in A-B$, $f^{-1}(\dot{u}) = [a,x_0)$, then $P_f(\dot{u}) = P[a,x_0) \triangleq P_0$

for any $u \notin A$, $f^{-1}(\dot{u}) = \phi$, then $P_f(\dot{u}) = P(\phi) = 0$.

Therefore falling shadow of the f is fuzzy set

$$\bar{A}(u) = \begin{cases} 1 & \text{if } u \in B \\ P_0 & \text{if } u \in A-B \\ 0 & \text{if } u \notin A \end{cases}$$

obviously, if $A = B$, then $A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$, i.e., $\bar{A} = A$.

Example 2-2: Suppose that $U = \{u_1, u_2, \ldots\ldots, u_n\}$

$A_i = \{u_i, u_{i+1}, \ldots\ldots, u_n\}$ $(i = 1,2,\ldots\ldots,n)$, and $I = [0,1]$.

Let mapping $f: I \rightarrow P(U)$ be that $f(0) = A_1 = U$,
if $\frac{i-1}{n} < x \leqslant \frac{i}{n}$, then $f(x) = A_i$,
so, $\dot{u}_i = \{A_1, A_2, \ldots\ldots, A_i\}$, $f^{-1}(\dot{u}_i) = [0, \frac{i}{n}]$ $(i = 1,2,\ldots\ldots, n)$

Therefore falling shadow of the f is fuzzy set

$$\bar{A}(u_i) = P(f^{-1}(\dot{u}_i)) = P([0,\frac{i}{n}]) \triangleq P_i \quad (i = 1,2,\ldots\ldots,n).$$

3. Estimation of the Function of Membership of Fuzzy Sets

In the following, I represents the real interval $[0,1]$.

Suppose that A is a fuzzy set on U, for any $y \in I$, write

$$H(y) = A_y = \{u \mid A(u) \geqslant y\}$$

then the H is called section-sets of the A.

It is apparent the H is an antitonic set-valued mapping from I to P(U).

From lamma 2-1, for any $u \in U$, there is $x = \sup\{y \mid y \in I, \ u \in H(y)\}$, such that

$$H^{-1}(\dot{u}) = \begin{cases} [0,x) & \text{if } u \notin H(x) \\ [0,x] & \text{if } u \in H(x) \end{cases}$$

Let probability p be uniform distribution on $(I, \mathbf{B})$, then falling shadow of the H is fuzzy set

$$\bar{A}(u) = P(H^{-1}(\dot{u})) = x \qquad (u \in U)$$

From the decomposition theorem of fuzzy sets we have

$$A(u) = \sup_{y \in I} \min(y, A_y(u)) = \sup\{y \mid y \in I, \ u \in A_y\}$$
$$= \sup\{y \mid y \in I, \ u \in H(y)\} = x \qquad (u \in U)$$

So, $A(u) = \bar{A}(u) \qquad (u \in U)$

From the above results we have

Theorem 3-1: Let A be a fuzzy set on U, then there is an antitonic set-valued mapping H from I to P(U) and probability P on the $(I, \mathbf{B})$, such that falling shadow of the H is the fuzzy set A.

From $A(u) = P(H^{-1}(\dot{u})) = P([0,x)) = P([0,x]) \quad (u \in U)$
it is thus clear that A(u) is equal to the probability taking $y \in I$ and $y<x$ uniformly.
here $x = \sup\{y \mid y \in I, \ u \in H(y)\}$

From lemma 2-1 $H^{-1}(\dot{u}) = \begin{cases} [0,x) & \text{if } u \notin H(x) \\ [0,x] & \text{if } u \in H(x) \end{cases}$
and $H^{-1}(\dot{u}) = \{y \mid y \in I, \ u \in H(y)\}$, we have

$$\{y \mid y \in I, \ u \in H(y)\} = \begin{cases} [0,x) & \text{if } u \notin H(x) \\ [0,x] & \text{if } u \in H(x) \end{cases}$$

We interpret the section-sets H of fuzzy set A as set-valued statistical experiment corresponding to I. For any $y \in I$, if $u \in H(y)$, we consider that the $H(y)$ has covered point u. If $u \notin H(y)$, we consider that the $H(y)$ has not covered point u. From "*", if $y < x$, then $H(y)$ has covered u. If $y > x$, then $H(y)$ has not covered u. It is the right the other way round. Therefore, the experiment taking $y \in I$ and $y < X$ correspond to the set-valued experiment H covered point u. Since the probability taking $y \in I$ and $y < x$ is $A(u)$, the probability of H that covers point u is also $A(u)$.

From law of great numbers, we make n-th experiments from the H independently, and have obtained n-sample $H_1, H_2, \ldots\ldots, H_n$ $(H_i \in P(U))$, for any $u \in U$, write

$$\bar{A}(u) = \frac{1}{n} \sum_{i=1}^{n} \chi_{H_i}(u)$$

so, $A(u) \approx \bar{A}(u)$.

On probability space $(I, \mathbf{B}, P)$ write

$$F(x) = P([0,x)) = P([0,x])$$

then the $F(x)$ is a distribution function.

so, $A(u) = P(H^{-1}(u)) = P([0,x)) = F(x)$

here $x = \sup\{\, y \mid y \in I, \ u \in H(y) \,\}$

Therefor, we can use the empirical distribution function with respect to $F(x)$ to estimate the function of membership of fuzzy set.

Example 3-1: Suppose that U is a set and $\{H_i\}_{i=1}^{n} \subset P(U)$, we might as well let $H_1 \supset H_2 \supset \ldots\ldots \supset H_n \neq \phi$.

Take $x_i \in I$ $(i = 1,2,\ldots\ldots, n)$ such that $x_1 \leq x_2 \leq \ldots\ldots \leq x_n$.

We make n-th set-valued experiments:

$$H(x_i) = H_i \qquad (i=1,2,\ldots\ldots,n).$$

so, obtain an empirical distributed function:

$$F_n^*(x) = \begin{cases} 0 & \text{if } x < x_1 \\ \frac{i}{n} & \text{if } x_i \leq x < x_{i+1} \\ 1 & \text{if } x \geq x_n \end{cases} \qquad (i=1,2,\ldots\ldots,n).$$

From W. Glivenko Theorem, we have

$$A(u) = F(x) \approx F_n^*(x) = \begin{cases} 0 & \text{if } x < x_1 \\ \frac{i}{n} & \text{if } x_i \leq x < x_{i+1} \\ 1 & \text{if } x \geq x_n \end{cases}$$

here $x = \sup\{x_i \mid 1 \leq i \leq n, u \in H(x_i)\}$

(See Fig. 1 & Fig. 2. on p.6)

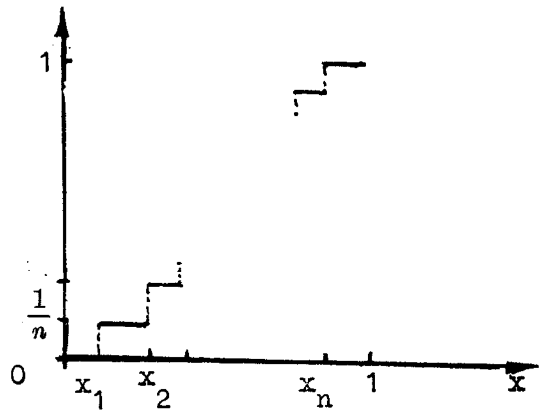

Fig. 1. $F_n^*(x)$

Fig 2. $\overline{A}(u)$

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