# EXTENSION OF CONSONANT BELIEF FUNCTIONS DEFINED ON AN ARBITRARY NONEMPTY CLASS OF L-FUZZY SETS*

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Abstract

The concepts of consonant belief function and B-consistency of set function defined on L-fuzzy sets are introduced and the results analogous to [1] are obtained.

Keywords: L-fuzzy set, Consonant belief function,
B-consistency of set function.

In this paper, let X be a nonempty set, $F_L(X) = \{ \underline{A};\ \underline{A}: X \longrightarrow L$, L is a complete lattice $\}$, and C* be an arbitrary nonempty subset of $F_L(X)$, $\mu$ be a mapping from C* into the unit interval [0 , 1], and we make the following convention: $\bigcap_{\emptyset}\{\cdot\} = X$, $\sup_{\emptyset}\{\mu(\cdot)\} = 0$, $\inf_{\emptyset}\{\mu(\cdot)\} = 1$.

<u>Definition 1.</u> A consonant belief function on $F_L(X)$ is a non-negative real valued set function $\beta : F_L(X) \longrightarrow [0\ ,\ 1]$ with the property:

$$\beta(\bigcap_{t\in T} \underline{A}_t) = \inf_{t\in T} \beta(\underline{A}_t), \text{ whenever } \{\underline{A}_t;\ t \in T\} \subset F_L(X),$$

where T is an arbitrary index set.

<u>Definition 2.</u> $\mu: C^* \longrightarrow [0, 1]$ is called B-consistent, if for every $\{\underline{A}_t;\ t \in T\} \subset C^*$, $\underline{A} \in C^*$, with $\underline{A} \supset \bigcap_{t\in T} \underline{A}_t$, we have

$$\mu(\underline{A}) \geq \inf_{t\in T} \mu(\underline{A}_t),$$

where T is an arbitrary index set.

<u>Theorem 1.</u> $\mu$ can be extended to a consonant belief function on $F_L(X)$, if and only if $\mu$ is B-consistent.

Proof. Necessity. Obvious.

Sufficiency. If we define

$$\beta: F_L(X) \longrightarrow [0, 1]$$

$$\underline{B} \longmapsto \inf_{x\in X} \sup_{\substack{(\bigcap_{s\in S^x} \underline{E}_s)(x) \leq \underline{B}(x) \\ \underline{E}_s \in C^*}} \inf_{s\in S^x} \mu(\underline{E}_s), \qquad (1)$$

where $S^x$ is an arbitrary index set, then $\beta$ is a consonant belief function on $F_L(X)$, and a extension of u on $C^*$. To conclude the assertions, we first prove that $\beta$ is a consonant belief function. In fact, the monotonicity of $\beta$ is obvious. By the monotonicity of $\beta$ , we have, for every $\{\underline{A}_t;\ t \in T\} \subset F_L(X)$,

$$\beta(\underline{A}_t) \geq \beta(\bigcap_{t\in T} \underline{A}_t),$$

and hence

$$\inf_{t\in T} \beta(\underline{A}_t) \geq \beta(\bigcap_{t\in T} \underline{A}_t),$$

where T is an arbitrary index set.

On the other hand, 1) when for every $x \in X$, $t \in T$ there exists $\{\underline{E}_s;\ s \in S^x\} \subset C^*$ such that

$$(\bigcap_{s\in S^x}\underline{E}_s)(x) \leq \underline{A}_t(x)$$

and for any $\varepsilon > 0$

$$\sup_{\substack{(\bigcap_{s\in S_t^x}\underline{E}_s)(x) \leq \underline{A}_t(x) \\ \underline{E}_s \in C^*}} \inf_{s\in S^x} \mu(\underline{E}_s) \leq \inf_{s\in S^x} \mu(\underline{E}_s) + \varepsilon,$$

then, since

$$(\bigcap_{t\in T}\bigcap_{s\in S_t^x}\underline{E}_s)(x) \leq \bigwedge_{t\in T}\underline{A}_t(x) = (\bigcap_{t\in T}\underline{A}_t)(x),$$

we have

$$\inf_{t\in T} \sup_{\substack{(\bigcap_{s\in S_t^x}\underline{E}_s)(x) \leq \underline{A}_t(x) \\ \underline{E}_s \in C^*}} \inf_{s\in S_t^x} \mu(\underline{E}_s) \leq \inf_{t\in T} \inf_{s\in S_t^x} \mu(\underline{E}_s) + \varepsilon$$

$$= \inf_{\substack{s\in\bigcap S_t^x \\ t\in T}} \mu(\underline{E}_s) + \varepsilon \leq \sup_{\substack{(\bigcap_{s\in S^x}\underline{E}_s)(x) \leq (\bigcap_{t\in T}\underline{A}_t)(x) \\ \underline{E}_s \in C^*}} \inf_{s\in S^x} \mu(\underline{E}_s) + \varepsilon,$$

this shows that

$$\inf_{x\in X}\inf_{t\in T} \sup_{\substack{(\bigcap_{s\in S_t^x}\underline{E}_s)(x) \leq \underline{A}_t(x) \\ \underline{E}_s \in C^*}} \inf_{s\in S_t^x} \mu(\underline{E}_s) \qquad (*)$$

$$\leq \inf_{x\in X} \sup_{\substack{(\bigcap_{s\in S^x}\underline{E}_s)(x) \leq (\bigcap_{t\in T}\underline{A}_t)(x) \\ \underline{E}_s \in C^*}} \inf_{s\in S^x} u(\underline{E}_s) + \varepsilon.$$

2) When there exists $x_o \in X$, $t_o \in T$, for every $\{\underline{E}_s;\ s \in S_{t_o}^{x_o}\} \subset C^*$ such that

$$(\bigcap_{s\in S_{t_o}^{x_o}}\underline{E}_s)(x_o) \nleq \underline{A}_{t_o}(x_o),$$

by using $\inf_{\emptyset}\{\mu(\cdot)\} = 1$, $\sup_{\emptyset}\{\mu(\cdot)\} = 0$, (*) is also true.

It yields that

$$\inf_{t\in T} \beta(\underline{A}_t) \leq \beta(\bigcap_{t\in T} \underline{A}_t).$$

Consequently,

$$\inf_{t\in T} \beta(\underline{A}_t) = \beta(\bigcap_{t\in T} \underline{A}_t),$$

which means $\beta$ is a consonant belief function.

Next, we prove that $\beta$ is an extension of u on C*. In fact, for every $\underline{B} \in C^*$, we have

$$\beta(\underline{B}) = \inf_{x\in X} \sup_{\substack{(\bigcap_{s\in S^x} \underline{E}_s)(x) \leq \underline{B}(x) \\ \underline{E}_s \in C^*}} \inf_{s\in S^x} \mu(\underline{E}_s) \geq \inf_{x\in X} \mu(\underline{B}) = \mu(\underline{B}).$$

On the other hand, for any $\varepsilon > 0$, every $x \in X$ there exists $\{\underline{E}_s;\ s\in S^x\} \subset C^*$ such that

$$\underline{B}(x) \geq (\bigcap_{s\in S^x} \underline{E}_s)(x) \geq (\bigcap_{s\in \bigcap_{x\in X} S^x} \underline{E}_s)(x),$$

and

$$\sup_{\substack{(\bigcap_{s\in S^x} \underline{E}_s)(x) \leq \underline{B}(x) \\ \underline{E}_s \in C^*}} \inf_{s\in S^x} \mu(\underline{E}_s) \leq \inf_{s\in S^x} \mu(\underline{E}_s) + \varepsilon,$$

hence, by using the B-consistence of $\mu$,

$$\beta(\underline{B}) = \inf_{x\in X} \sup_{\substack{(\bigcap_{s\in S^x} \underline{E}_s)(x) \leq \underline{B}(x) \\ \underline{E}_s \in C^*}} \inf_{s\in S^x} \mu(\underline{E}_s)$$

$$\geq \inf_{x\in X} \inf_{s\in S^x} \mu(\underline{E}_s) + \varepsilon = \inf_{s\in \bigcap_{x\in X} S^x} \mu(\underline{E}_s) + \varepsilon \leq \mu(\underline{B}) + \varepsilon,$$

therefore

$$\beta(\underline{B}) \leq \mu(\underline{B}).$$

Consequently,

$$\beta(\underline{B}) = \mu(\underline{B}),$$

and we complete the proof of the theorem.

In usual case, the extension of a mapping $\mu$ with B-consistent from an arbitrary nonempty class of the L-fuzzy subsets of X into the unit interval $[0, 1]$ to a consonant belief function on $F_L(X)$ may not be unique. All extensions of consonant belief function is denoted $E_\beta(\mu)$. By using theorem 1, we know that $E_\beta(\mu)$ is nonempty, if $\mu$ is B-consistent.

For two mappings $\mu_1$: $F_L(X) \longrightarrow [0, 1]$ and $\mu_2$: $F_L(X) \longrightarrow [0, 1]$, we define ordering relation " $\leq$ ":

$\mu_1 \leq \mu_2$ if and only if $\mu_1(\underline{A}) \leq \mu_2(\underline{A})$, for every $\underline{A} \in F_L(X)$.

It is easy to prove that " $\leq$ " is a partial ordering relation on $E_\beta(\mu)$. Therefore the greatest lower bound of $\mu_1$, $\mu_2 \in E_\beta(\mu)$ can be defined by

$$(\inf\{\mu_1 , \mu_2\})(\underline{A}) = \mu_1(\underline{A}) \wedge \mu_2(\underline{A}), \text{ for all } \underline{A} \in F_L(X).$$

Theorem 2. $(E_\beta(\mu) , \leq)$ is a lower semi-lattice, and the extension $\beta$ defined by (1) is the least element of $E_\beta(\mu)$.

Proof. 1) Obviously, $(E (u) , \leq)$ is a lower semi-lattice.

2) The extension $\beta$ defined by (1) is the least element of the $E_\beta(\mu)$. For arbitrary $\beta' \in E_\beta(\mu)$, $\underline{B} \in F_L(X)$, we define

$$\underline{F}_x(y) = \begin{cases} \sup\limits_{\substack{(\bigcap_{s\in S^x}\underline{E}_s)(x) \leq \underline{B}(x) \\ \underline{E}_s \in C^*}} (\bigcap_{s\in S^x}\underline{E}_s)(x) & \text{if } y = x; \\ 0 & \text{if } y \neq x, \end{cases}$$

for every $x \in X$. If $\underline{B}(x) \geq (\bigcap_{s\in S^x}\underline{E}_s)(x)$, $\underline{E}_s \in C^*$, we have

$$\underline{F}_x \supset \bigcap_{s\in S^x} \underline{E}_s,$$

hence

$$\beta'(\underline{F}_x) \geq \inf_{s\in S^x} \beta'(\underline{E}_s),$$

therefore

$$\beta'(\underline{F}_x) \geq \sup_{\substack{(\bigcap_{s\in S^x}\underline{E}_s)(x) \leq \underline{B}(x) \\ \underline{E}_s \in C^*}} \inf_{s\in S^x} \beta'(\underline{E}_s)$$

$$= \sup_{\substack{(\bigcap_{s\in S^x}\underline{E}_s)(x) \leq \underline{B}(x) \\ \underline{E}_s \in C^*}} \inf_{s\in S^x} \mu(\underline{E}_s),$$

for every $x \in X$, it follows, by using

$$\underline{B}(x) \geq \sup_{\substack{(\bigcap_{s\in S^x}\underline{E}_s)(x) \leq \underline{B}(x) \\ \underline{E}_s \in C^*}} (\bigcap_{s\in S^x}\underline{E}_s)(x) = \underline{F}_x(x) \geq (\bigcap_{x\in X}\underline{F}_x)(x),$$

that

$$\beta(\underline{B}) = \inf_{x\in X} \sup_{\substack{(\bigcap_{s\in S^x}\underline{E}_s)(x) \leq \underline{B}(x) \\ \underline{E}_s \quad C^*}} \inf_{s\in S^x} \mu(\underline{E}_s) \leq \inf_{x\in X} \beta'(\underline{F}_x)$$

$$= \beta'(\bigcap_{x\in X}\underline{F}_x) \leq \beta'(\underline{B}).$$

References


[1] Wang Zhenyuan, Extension of Consonant Belief Functions Defined on an Arbitrary Nonempty Class of Sets, Fuzzy Systems and Math Mathematics, 1(1987), 66-70.

[2] Liu Guiting and Wang Zhenyuan, Extension Theorem of Consonant Belief Function, Kexue Tongbao, 9(1985), 718.

[3] G. Banon, Distinction Between Several Subsets of Fuzzy Measures, Fuzzy Sets and Systems, 5(1981), 291-305.

[4] Zhang Guangquan, Semi-lattice Structure of All Extensions of Possibility Measure and Consonant Belief Function on the Fuzzy Set, BUSEFAL, 33(1987), 95-103.

[5] Zhang Guangquan, The Extensions of A Class of Sc-fuzzy Measure, Hebei University, 3(1987), 11-17.

[6] G. Shafer, A Mathematical Theory of Evidence, Princetion University Press, Princeton, NJ, 1976.

[7] Wang Zhenyuan and Zhang Zhipeng, On the Extension of Possibity Measures, BUSEFAL, 18(1984), 26-32.

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