A CLASS OF BOUNDS ON PROBABILITY OF ERROR BASED ON INFORMATION ENERGY FOR THE FUZZY DISCRIMINATION PROBLEM

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SUMMARY

In this paper in order to stablish bounds for average probabity in the fuzzy discrimination problem the "Information Energy on fuzzy classes and exact information" the "Information Energy on exact classes and fuzzy information" and the "Information Energy on fuzzy classes and fuzzy information" are considered. These concepts are inspired in the measure of information "Information Energy" introduced in Information Theory by Onicescu (1966).

1.- INTRODUCTION

Consider the classical discrimination problem which is concerned with the assignement of a given object to one of n known classes of its observed characteristics . Let $C=\{C_i / i=1,\dots,n\}$ be a discrete random variable , associated with a finite set of hypothesis H_i. The uncertainty about the true class is expressed by the prior probability vector $P(C)=(P(C_1),\dots,P(C_n))$, where $P(C_i)$ represents the prior probability of the class C_i, $P(C_i)\geq 0$ and $\sum_{i=1}^{n} P(C_i)=1$. Let x be an observation on the probabilistic information system X, where X is distributed according to one of n possible conditional density functions $f(x/C_i)$, when C_i is the true class. These conditional probabilities , along with the prior distribution on C, determine: The joint probability distribution, $p(C_i,x)$, $(i=1,\dots,n,\ x\in X)$; the marginal probability disribution on X, $f(x)$ and the posterior distribution on C, $p(C_i/x)$ $(i=1,\dots,n)$.

It is known that the decision rule which minimizes the probability of error is the Bayes decision rule which assigns x the class with the highest a posterior probability. This rule leads to a probability of error which is given by $1-\max_i p(C_i/x)$ .Prior to observing X, the probability of error, P_e, associated with X is defined as $P_e = 1-E_X(\max_i p(C_i/x))$, where E_X is the expectation with respect to X.

Tanaka,, Okuda and Asai (13) formulate the discrimination problem with fuzzy classes and fuzzy information using the probability of fuzzy events and derive a bound for the average error probability , when the decision in the classifier is made according to the fuzzified Bayes method . They consider the following cases: (a) Fuzzy classes and exact information, (b) Exact classes and fuzzy information (c) Fuzzy classes and fuzzy information.

In this paper , in order to stablish a bound for average error probability, when the decision in the classifier is made according to the fuzzified Bayes method, the " Information Energy in fuzzy classes and exact information", the "Information Energy in exact classes and fuzzy information" and the "Information Energy in fuzzy classes and fuzzy information" are considered. These concepts are inspired in the measure of information "Information Energy " introduced in Information Theory by Onicescu (7) but they have a meaning quite different from the one of Onicescu's Information Theory, since they integrate on the one hand the information in a moment before carrying out an experiment and on the other hand the uncertainty of meaning of fuzzy sets which it is expressed by the membership function.

2.- EXACT CLASSES AND FUZZY INFORMATION

In this section we consider the discrimination problem, when we have exact classes $C=\{C_1,\ldots,C_n\}$ and the available experimental information on which these conclusions will be based is not exact, but rather it may be described by means of fuzzy events on the space X; i.e., we asume that the ability to observe the experimental outcome only allows the statistician to assimilate each elementary observable event with fuzzy information (14), where

Definition 2.1

A fuzzy information $\mathcal{X}$ from X is a fuzzy event on X which is characterized by a Borel-measurable membership function $\mu_{\mathcal{X}}$ from X to [0,1] where $\mu_{\mathcal{X}}(x)$ represents the "grade of compatibility of x in $\mathcal{X}$.

In the definition of measures of information associated with an experiment when the available experimental information is given by grouping of experimental observations, the set of all elementary events associated with the experiment is a classical partition of the sample space. In a similar way and for the sake of operativeness, we will hereafter assume that the available "elementary observable events" determine a partition of fuzzy sets on the sample space or "fuzzy partition" which is called fuzzy information system according to the notion introduced by Tanaka et al. (14):

Definition 2.2

A fuzzy information system $\mathcal{X}^*$, associated with X is a fuzzy partion (orthogonal system) of X by means of fuzzy informations $\mathcal{X}$ from $\mathcal{X}^*$, that is, $\sum_{\mathcal{X}\in\mathcal{X}^*} \mu_{\mathcal{X}}(x) = 1, \ \forall x \in X$

As the attention is focussed on the classe C_i governing the distribution of the exact information from X, but the present available information is fuzzy, it would be interesting to obtain the probabilistic definition stated by Zadeh (15) as follows.

Definition 2.3

The probabilty of fuzzy information $\mathcal{X}$ given the classe C_i is defined by

$$\mathcal{P}(\mathcal{X}/C_i) = \int_X \mu_{\mathcal{X}}(x)\ f(x/C_i)\ d\tau(x)$$

As we assume the existence of a prior probability on C we can introduce the following probability distributions:

Definition 2.4

The marginal probability distribution on $\mathcal{X}^*$ of the fuzzy information $\mathcal{X}$ is given by

$$\mathcal{P}(\mathcal{X}) = \int_X \mu_{\mathcal{X}}(x)\ f(x)\ d\tau(x)$$

Definition 2.5

The posterior probability distribution on C given $\mathcal{X} \in \mathcal{X}^*$ is given by

$$\mathcal{P}(C_i/\mathcal{X}) = \frac{\int_X \sum_{i=1}^{n} \mu_{\mathcal{X}}(x)\ f(x/C_i)\ d\tau(x)}{\mathcal{P}(\mathcal{X})}$$

Let us consider the problem of classifying a fuzzy observation $\mathcal{X}$ as coming from one of n possible classes $C_1,..,C_n$. In this case , the fuzzified Bayes method consists in evaluating the posterior probability $\mathcal{P}(C_i/\mathcal{X})$ and assigning the fuzzy observation $\mathcal{X}$ to the class C_r, $1 \le r \le n$ for which $\mathcal{P}(C_i/\mathcal{X})$ is maximal. This decision rule leads to a a probability of error given by

$$P_e = 1 - \sum_{\mathcal{X}\in\mathcal{X}^*} \max_i \mathcal{P}(C_i/\mathcal{X})\ \mathcal{P}(\mathcal{X})$$

Now we stablish a bound for P_e .First, we stablish the following definitions :

Definition 2.6

The Conditional Information Energy of C given by the fuzzy information $\mathcal{X}$ is

$$\mathcal{E}\ (\mathcal{P}(C/\mathcal{X})) = \sum_{i=1}^{n} \Big(\mathcal{P}(C_i/\mathcal{X}) \Big)^2$$

Definition 2.7

The Conditional Information Energy of C given the fuzzy Information system $\mathcal{X}^*$ is

$$\mathcal{E}\ (\mathcal{X}^*, C) = \sum_{\mathcal{X}\in\mathcal{X}^*} \mathcal{E}(\ \mathcal{P}(C/\mathcal{X}))\ \ \mathcal{P}(\mathcal{X})$$

Now, we present a theorem giving upper and lower bounds on probability of error.

Theorem 2.1

Let $\mathcal{E}(\mathcal{X}^*,C)$ be the conditional information energy of C given the fuzzy information system $\mathcal{X}^*$. Then it is verified

$$(1/2)(1-\mathcal{E}(\mathcal{X}^*,C) \leq P_e \leq (1-\mathcal{E}(\mathcal{X}^*,C))$$

Proof

First we stablish that $P_e \leq 1-\mathcal{E}(\mathcal{X}^*,C)$. In fact

$$1-\mathcal{E}(\mathcal{X}^*,C) = 1-\sum_{\mathcal{X}\in\mathcal{X}^*}\sum_{i=1}^{n}\left[\mathcal{P}(C_i/\mathcal{X})\right]^2\mathcal{P}(\mathcal{X}) =$$

$$= \sum_{\mathcal{X}\in\mathcal{X}^*}\left[1-\sum_{i=1}^{n}\left(\mathcal{P}(C_i/\mathcal{X})\right)^2\mathcal{P}(\mathcal{X})\right] \geq \sum_{\mathcal{X}\in\mathcal{X}^*}\left(1-\max_i \mathcal{P}(C_i/\mathcal{X})\right)\mathcal{P}(\mathcal{X}) = P_e$$

It is easy to prove that $0\leq\mathcal{E}(\mathcal{X}^*,C)\leq 1$. If $0\leq x\leq 1$, it follows that $x+1\leq 2x^{1/2}$, hence $(1/2)(1-x)\leq 1-x^{1/2}$ (1)

From inequality (1) it follows that

$$(1/2)(1-\mathcal{E}(\mathcal{X}^*,C)) \leq 1-\left[\mathcal{E}(\mathcal{X}^*,C)\right]^{1/2}$$

If we prove that

$$1-\left[\mathcal{E}(\mathcal{X}^*,C)\right]^{1/2} \leq P_e$$

we obtain the result of theorem 2.1. First we note that

$$\max_i \mathcal{P}(C_i/\mathcal{X}) \leq \left[\sum_{i=1}^{n}\left(\mathcal{P}(C_i/\mathcal{X})\right)^2\right]^{1/2} = \left[\mathcal{E}(\mathcal{P}(C/\mathcal{X}))\right]^{1/2} \quad (2)$$

Multiply both sides of inequality (2) by $\mathcal{P}(\mathcal{X})$ and summing on both sides on $\mathcal{X}^*$, we have

$$1 - P_e \leq \sum_{\mathcal{X}\in\mathcal{X}^*}\mathcal{E}\,(\mathcal{P}(C/\mathcal{X}))^{1/2}.\mathcal{P}(\mathcal{X})$$

Now, let Z be a discrete random variable taking on the values $\mathcal{E}\,(\mathcal{P}(C/\mathcal{X}))$ with probabilities $\mathcal{P}(\mathcal{X})$ and consider the convex function $g(x)= x^{1/2}$, then

$$\cdot \quad E(Z) = \sum_{\mathcal{X}\in\mathcal{X}^*} \mathcal{E}\,(\mathcal{P}(C/\mathcal{X}))\;.\;\mathcal{P}(\mathcal{X})$$

$$\cdot \quad g(E(Z)) = \left(\mathcal{E}\,(\mathcal{P}(C/\mathcal{X}) \right)^{1/2}$$

We may now aply Jensen's inequality to obtain

$$1 - P_e \le \left(\mathcal{E}\,(\mathcal{X}^*, C) \right)^{1/2}$$

Tanaka, Okuda and Asai prove the following relation : $P_e \le \mathcal{H}(\mathcal{X}^*, C)$, where,

$$\mathcal{H}(\mathcal{X}^*, C) = - \sum_{\mathcal{X}\in\mathcal{X}^*} \sum_{i=1}^{n} \mathcal{P}(C_i/\mathcal{X}) \log_a \mathcal{P}(C_i/\mathcal{X})\; P(\mathcal{X})$$

In the following theorem, we study the relation between $\mathcal{H}(\mathcal{X}^*, C)$ and $\mathcal{E}(\mathcal{X}^*, C)$.

Theorem 2.2

$$\log_a e\; P_e \le (\; 1 - \mathcal{E}(\mathcal{X}^*, C)\;)\; \log_a\, e \le \mathcal{H}(\mathcal{X}^*, C)$$

Proof

The function $g(x)=\log_a x$ is strictly concave on $(0,\infty)$. The line tangent to $\log_a x$ at $x=1$ is $(x-1)\log_a e$ and verifies

$$(x-1)\log_a e \ge \log_a x \quad (1)$$

and the inequality being strict for all $x\ne1$. Now put $x=\mathcal{P}(C_i/\mathcal{X})$, we have

$$\mathcal{P}(C_i/\mathcal{X})-1 \ge \log_a \mathcal{P}(C_i/\mathcal{X})\; (\log_a\, e)^{-1} \quad (2)$$

multiply both sides inequality (2) by $-\mathcal{P}(C_i/\mathcal{X})$, it follows

$$-\left[\mathcal{P}(C_i/\mathcal{X})\right]^2 + \mathcal{P}(C_i/\mathcal{X}) \geq -\mathcal{P}(C_i/\mathcal{X}) \log_a \mathcal{P}(C_i/\mathcal{X}) \, (\log_a e)^{-1} \quad (3)$$

summing this last inequality on both sides over i, we have

$$1-\sum_{i=1}^{n} \left[\mathcal{P}(C_i/\mathcal{X})\right]^2 \leq -\sum_{i=1}^{n} \mathcal{P}(C_i/\mathcal{X}) \log_a \mathcal{P}(C_i/\mathcal{X}) \, (\log_a e)^{-1} \quad (4)$$

Multiply both sides of inequality (4) by $\mathcal{P}(\mathcal{X})$ and summing on both sides on $\mathcal{X}^*$, we obtain

$$\log_a e(1-\mathcal{E}(\mathcal{X}^* C) \leq H(\mathcal{X}^*, C)$$

On the other hand, by theorem 2.1 it is verified that

$$P_e \leq 1 - \mathcal{E}(\mathcal{X}^*, C)$$

then

$$\log_a e \; P_e \leq (1- \mathcal{E}(\mathcal{X}^*, C)) \log_a e$$

3 .- FUZZY CLASSES AND EXACT INFORMATION

In this section we deal with the discrimination problem when the available experimental information is exact and we have fuzzy classes. It is reasonable that we consider the fuzzy class set $\mathcal{C}^*$ consisted of all $\mathcal{C}^i$, rather than the set of exact classes C, where each $\mathcal{C}^i$ is expressed by our interested words on C . Here a fuzzy class $\mathcal{C}^i$ is defined by a membership function $\mu_{\mathcal{C}^i}(C_j)$, whose value means the grade of compatibility of C_j on $\mathcal{C}^i$, and it is assumed that

$$\sum_{i=1}^{k} \mu_{\mathcal{C}^i}(C_j) = 1 \qquad \forall \; j=1,..,n$$

In this case, the fuzzified Bayes method consists in evaluating the posterior probability $\mathcal{P}(\mathcal{C}^i/x)$ and assigning the observation x to the class $\mathcal{C}^r$, $1\leq r \leq k$, for which $\mathcal{P}(\mathcal{C}^i/x)$ is maximal where $\mathcal{P}(\mathcal{C}^i/x)$ is given by

$$\mathcal{P}(\mathcal{C}^i/x) = \frac{1}{f(x)} \sum_{j=1}^{k} \mu_{\mathcal{C}^i}(C_j) \, f(x/C_j) \, p(C_j)$$

This decision rule leads to an average probability of error given by

$$P_e^* = 1 - \int_X \text{máx } \mathcal{P}(\mathcal{C}^i/x)\ f(x)\ d\tau(x)$$

Now we stablish bounds for P_e^*. In order to do so, let us introduce the Conditional Information Energy concept of $\mathcal{C}$.

Definition 3.1

The Conditional Information Energy of $\mathcal{C}^*$ given the observation x is

$$\mathcal{E}(x,\mathcal{C}^*) = \sum_{i=1}^{k} \left(\mathcal{P}(\mathcal{C}^i/x)^2 + \mathcal{P}(\bar{\mathcal{C}}^i/x)^2 \right)$$

Definition 3.2

The Conditional Information Energy of $\mathcal{C}^*$ given the probabilistic information system X, is

$$\mathcal{E}(X,\mathcal{C}^*) = \int_X \sum_{i=1}^{k} \left(\mathcal{P}(\mathcal{C}^i/x)^2 + \mathcal{P}(\bar{\mathcal{C}}^i/x)^2 \right) f(x)\ d\tau(x)$$

Pardo et al. (12), explaining the meaning of the "Informational Energy of the fuzzy state space ", emphasize that it incorporates fuzziness and randomness. Nevertheless, this measure does not try to generalize Onnicescu's Informational Energy but rather it generalizes, in certain a sense the nonprobabilistic non fuzziness measures (Pardo (8)).

Now, we present a theorem giving upper and lower bounds of probability of error.

Theorem 2.1

Let $\mathcal{E}(X,\mathcal{C}^*)$ be the conditional information energy of $\mathcal{C}^*$ given the probabilistic information system X, then bounds by the average probability of error, P_e^*, associated with X are given by

$$(1/k)\left[k-1-\left(1-(k-1)^3-k(k-1)\mathcal{E}(X,\mathcal{C}^*)\right)^{1/2}\right] \leq P_e^* \leq \left(k-\mathcal{E}(X,\mathcal{C}^*)\right)\ (1/2)$$

Proof

It is easy to prove that

$$\sum_{i=1}^{k}\left(\mathcal{P}(\mathcal{C}^i/x)^2 + \mathcal{P}(\bar{\mathcal{C}}^i/x)^2\right) \leq 2 \max_i \mathcal{P}(\mathcal{C}^i/x)^2 + k - 2 \qquad (1)$$

Taking the expectation with respect to x on both sides of inequality (1) yields

$$\mathcal{E}\ (X,\mathcal{C}^*) \leq 2\ (1-P_e^*\ ) + K - 2$$

hence

$$P_e^* \leq 1/2\left(k - \mathcal{E}\ (X,\mathcal{C}^*)\ \right)$$

On the other hand, if we denote

$$\mathcal{P}(\mathcal{C}^r/x) = \max_i \mathcal{P}(\mathcal{C}^i/x)$$

it follows that

$$\sum_{i=1}^{k}\left(\mathcal{P}(\mathcal{C}^i/x)^2 + \mathcal{P}(\bar{\mathcal{C}}^i/x)^2 \right) = \mathcal{P}(\mathcal{C}^r/x)^2 + \sum_{\substack{i=1\\ i\neq r}}^{k} \mathcal{P}(\mathcal{C}^i/x)^2 + \sum_{i=1}^{k}\left(1 - \mathcal{P}(\mathcal{C}^i/x)\right)^2$$

Using the following inequalities

$$\sum_{i=1}^{k}\left(1 - \mathcal{P}(\mathcal{C}^i/x) \right)^2 \geq \frac{(k-1)^2}{k}$$

$$\sum_{\substack{i=1\\ i\neq r}}^{k} \mathcal{P}(\mathcal{C}^i/x)^2 \geq \frac{(1 - \mathcal{P}(\mathcal{C}^r/x))^2}{k-1}$$

we have the following inequality

$$\sum_{i=1}^{k}\left(\mathcal{P}(\mathcal{C}^i/x)^2 + \mathcal{P}(\bar{\mathcal{C}}^i/x)^2 \right) \geq \mathcal{P}(\mathcal{C}^r/x)^2 + \frac{(k-1)^2}{k} + \frac{(1 - \mathcal{P}(\mathcal{C}^r/x))^2}{k-1}$$

If we denote $y = \mathcal{P}(\mathcal{C}^r/x)$, we obtain

$$\mathcal{E}\ (x,\mathcal{C}^*) \geq \frac{y^2\ k\ (k-1) + (k-1)^3 + (1-y)^2 k}{k\ (k-1)} = \frac{y^2\ k^2 - 2ky + k + (k-1)^3}{k\ (k-1)}$$

Hence,

$$k^2\ y^2 - 2\ k\ y + k + (k-1)^3 - k\ (k-1)\ \mathcal{E}(x,\mathcal{C}^*) \leq 0$$

Solving the last inequation we derive

$$1 - \max_i \mathcal{P}(\mathcal{C}^i/x) \geq \frac{1}{k}\left(k-1 - \left(1-(k-1)^3 - k(k-1)\ \mathcal{E}(x,\mathcal{C}^*)\right)^{1/2}\right) \quad (2)$$

If we take the expectation with respect to x on both sides of inequality (2), it follows from Jensen's inequality that

$$P_e^* \geq \frac{1}{k}\left(k-1 - \left(1 -(k-1)^3 - k -k\ (K-1)\ \ \mathcal{E}\ (X,\mathcal{C}^*)\right)^{1/2}\right)$$

Tanaka, Okudaa and Asai (14) prove the following relation

$$P_e^* \leq \mathcal{H}(X,\mathcal{C}^*)$$

where,

$$\mathcal{H}(X,\mathcal{C}^*) = - \int_X \left(\sum_{i=1}^{k}\left(\mathcal{P}(\mathcal{C}^i/x)\ \log_a\ \mathcal{P}(\mathcal{C}^i/x)+\mathcal{P}(\bar{\mathcal{C}}^i/x)\ \log_a \mathcal{P}(\bar{\mathcal{C}}^i/x)\right) f(x)d\tau(x)\right.$$

Now we stablish the relationship between $\mathcal{E}(X,\mathcal{C}^*)$ and $\mathcal{H}(X,\mathcal{C}^*)$.

Theorem 3.2

$$2\ P_e^*\ \log_a e \leq (k - \mathcal{E}(X,\mathcal{C}^*))\ \log_a e \leq \mathcal{H}(X,\mathcal{C}^*)$$

Proof

By inequality (1) of theorem 2.1, now we can stablish that

$$\mathcal{P}(\mathcal{C}^i/x) - 1 \geq \log_a\ \mathcal{P}(\mathcal{C}^i/x)\ (\log_a e)^{-1} \quad (1)$$

$$\mathcal{P}(\bar{\mathcal{C}}^i/x) - 1 \geq \log_a\ \mathcal{P}(\bar{\mathcal{C}}^i/x)\ (\log_a e)^{-1} \quad (2)$$

Multiply both sides of inequality (1) by $-\mathcal{P}(\mathcal{C}^i/x)$ it follows

$$- \mathcal{P}(\mathcal{C}^i/x)^2 + \mathcal{P}(\mathcal{C}^i/x) \leq - \mathcal{P}(\mathcal{C}^i/x)\ \log_a\ \mathcal{P}(\mathcal{C}^i/x)\ (\log_a\ e)^{-1} \quad (3)$$

Similarly we have,

$$- \mathcal{P}(\bar{\mathcal{C}}^i/x)^2 + \mathcal{P}(\bar{\mathcal{C}}^i/x) \leq - \mathcal{P}(\bar{\mathcal{C}}^i/x)\ \log_a\ \mathcal{P}(\bar{\mathcal{C}}^i/x)\ (\log_a e)^{-1} \quad (4)$$

Summing these last inequalities on both sides over i we obtain

$$1 - \sum_{i=1}^{k} \mathcal{P}(\mathcal{C}^i/x)^2 \leq - \sum_{i=1}^{k} \mathcal{P}(\mathcal{C}^i/x) \log_a \mathcal{P}(\mathcal{C}^i/x) \ (\log_a e) \quad (5)$$

$$k- 1 - \sum_{i=1}^{k} \mathcal{P}(\bar{\mathcal{C}}^i/x)^2 \leq - \sum_{i=1}^{k} \mathcal{P}(\bar{\mathcal{C}}^i/x) \log_a \mathcal{P}(\bar{\mathcal{C}}^i/x) \ (\log_a e) \quad (6)$$

Summing inequalities (5) and (6) we have

$$k - \sum_{i=1}^{k} \left(\mathcal{P}(\mathcal{C}^i/x)^2 + \mathcal{P}(\bar{\mathcal{C}}^i/x)^2 \right) \leq$$

$$- (\log_a e)^{-1} \left(\sum_{i=1}^{k} \mathcal{P}(\mathcal{C}^i/x) \log_a \mathcal{P}(\mathcal{C}^i/x) + \sum_{i=1}^{k} \mathcal{P}(\bar{\mathcal{C}}^i/x) \log_a \mathcal{P}(\bar{\mathcal{C}}^i/x) \right) \quad (7)$$

Taking the expectation with respect to x on both sides of inequality (7) we obtain

$$\log_a e \ (k - \mathcal{E} \ (X, \mathcal{C}^*) \leq \mathcal{H}(X, \mathcal{C}^*)$$

On the other hand , by theorem 3.1 it is verifies

$$2 \ P_e^* \leq k - \mathcal{E}(X, \mathcal{C}^*)$$

then

$$2 \ P_e^* \ \log_a e \leq (\ k - \mathcal{E}(X, \mathcal{C}^*) \ ) \ \log_a e$$

The conceps and results established in the precedent sections allow us to extend notions and results to the case "Fuzzy classes and fuzzy information" . Hovever, due to the analogy of this case with the precedent cases we not include the development.

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