ERGODIC THEOREMS ON FUZZY QUANTUM SPACES

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In this paper, we present a generalization of the individual ergodic theorem. It deals with a continuation of results of ergodic theory for fuzzy quantum spaces [3].

Now, we introduce the notions which we shall use in the following.

By a <u>fuzzy quantum space</u> we understand a couple (X,M), where X is a nonempty set and $M \subset [0,1]^X$ such that the following conditions are satisfied:

(i) if $[1]_X(x) = 1$ for any $x \in X$, then $[1]_X \in M$;

(ii) if $a \in M$, then $a^{\perp} := 1 - a \in M$;

(iii) if $[1/2]_X(x) = 1/2$ for any $x \in X$, then $[1/2]_X \notin M$;

(iv) $\bigcup_{n=1}^{\infty} a_n := \sup_n a_n \in M$, for any $\{a_n\}_{n=1}^{\infty} \subset M$:

By $\bigcap_n a_n$ we mean $\inf_n a_n$.

An <u>F-state</u> of a fuzzy quantum space (X,M) is a mapping $m: M \longrightarrow [0,1]$ such that

(i) $m(a \cup (1 - a)) = 1$ for every $a \in M$;

(ii) if $a_i \in M$ $(i = 1, 2, \dots)$ and $a_i \leq 1 - a_j$ $(i \neq j)$ then

$$m(\bigcup_i a_i) = \sum_i m(a_i).$$

In the fuzzy set theory, the mapping m is called a P-measure and M is a fuzzy soft σ-algebra ([7]).

An <u>F-observable</u> on a fuzzy quantum space (X,M) is a mapping $x: B(R^1) \longrightarrow M$ satisfying the following properties:

(i) $x(E^c) = 1 - x(E)$ for every $E \in B(R^1)$;

(ii) if $\{E_n\}_{n=1}^{\infty} \subset B(R^1)$, then $x(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} x(E_n)$, where $B(R^1)$ is the Borel σ-algebra of the real line R^1, and E^c denotes the complement of the set E in R^1. For an F-observable x we put $B_x(t) = x((-\infty, t))$, $t \in R^1$, and any F-observable is uniquely determined by the system $\{B_x(t): t \in R^1\}$ [4]. We define a <u>question observable</u> x_a of a fuzzy set $a \in M$ as a mapping from $B(R^1)$ into M such that

$$x_a(E) = \begin{cases} a \cap a^\perp & \text{if } 0,1 \notin E \\ a^\perp & \text{if } 0 \in E,\ 1 \notin E \\ a & \text{if } 0 \notin E,\ 1 \in E \\ a \cup a^\perp & \text{if } 0,1 \in E, \end{cases} \quad \text{for any } E \in B(R^1).$$

It is evident that x_a plays the role of the <u>indicator</u> of the fuzzy set $a \in M$. The question observable of the null fuzzy set 0 we denote by σ, i.e., $\sigma = x_0$.

If $f: R^1 \to R^1$ is a Borel measurable function, then $f \circ x: E \mapsto x(f^{-1}(E))$, $E \in B(R^1)$ is an F-observable of (X,M). The <u>spectrum</u> of an F-observable x we mean the set $\sigma(x) = \bigcap\{C \subset R^1: C$ is closed and $x(C) = x(R^1)\}$. An F-observable x is bounded if $\sigma(x)$ is a bounded set, in this case, we define the <u>norm</u> of x, $\|x\|$, via $\|x\| = \sup\{|t|: t \in \sigma(x)\}$. In [4], it has been defined the sum of any pair x and y of F-observables of (X,M) as follows:

By the <u>sum</u> of any pair of two F-observables x and y we mean a unique F-observable x + y for which we have

$$B_{x+y}(t) = \bigcup_{r \in Q} (B_x(r) \cap B_y(t - r)), \quad t \in R^1,$$

where Q is the set of all rationals in R^1.

The sum of two F-observables exists always ([4],[5]), and it coincides with pointwisely defined sum of observables for a σ-algebra of crisp subsets.

The difference of x and y is defined as $x - y = x + (-y)$, where $(-y)(E) = y(\{t: -t \in E\})$, $E \in B(R^1)$.

If x is an F-observable and m is an F-state, then the <u>mean value</u> of x in m we shall defined as follows

$$m(x) = \int_{R^1} t \, dm_x(t) := \int x \, dm, \text{ if the integral exists and}$$

is finite, where m_x is a probability measure on $B(R^1)$ defined via $m_x(E) = m(x(E))$, $E \in B(R^1)$.

A mapping $\tau: M \to M$ such that

(i) $\tau(a^{\perp}) = \tau(a)^{\perp}$, $a \in M$;

(ii) $\tau(\bigcup_{i=1}^{\infty} a_i) = \bigcup_{i=1}^{\infty} \tau(a_i)$, $\{a_i\}_{i=1}^{\infty} \subset M$ is called a <u>homomorphism</u> of (X,M). We say that a homomorphism τ of (X,M) is <u>invariant</u> in an F-state m if $m(\tau(a)) = m(a)$, $a \in M$. A homomorphism τ of (X,M) invariant in an F-state m is said to be <u>ergodic in m</u> if the statement $m(a \cap \tau(a^{\perp})) = 0 = m(\tau(a) \cap a^{\perp})$ implies $m(a) \in \{0,1\}$. It τ is a homomorphism and x is an F-observable, then $\tau \circ x$: $E \longmapsto (x(E))$, $E \in B(R^1)$, is an F-observable of (X,M), too.

Now, we recall that the sequence $\{x_n\}_{n=1}^{\infty}$ of F-observables of a fuzzy quantum space (X,M) <u>converges</u> to an F-observable x <u>almost everywhere</u> in an F-state m (in short $x_n \longrightarrow x$ a.e. [m]) if for every $\varepsilon > 0$

$$m(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} ((x_n - x)([-\varepsilon,\varepsilon]))) = 1.$$

ERGODIC THEOREMS

In this part, we generalize some results of R. Mesiar [6] for fuzzy quantum space.

Let us define, according to [1], $I_o = \{a \in M: \exists c \geqslant 1/2,\ c \in M$, such that $a \cap c \leqslant 1/2\}$, $I_m = \{a \in M: m(a) = 0\}$, then $I_o \subseteq I_m$ and I_o (I_m) is a σ-ideal, that is

(i) if $a \in M$, $b \in I_o$, $a \leqslant b$ then $a \in I_o$;

(ii) if $\{a_i\} \subset I_o$, then $\bigcup_i a_i \in I_o$;

(iii) $a \cap a^{\perp} \in I_o$ for every $a \in M$;

(iv) if $a \cap c \in I_o$ for some $c \geqslant 1/2$, $c \in M$, then $a \in I_o$.

We define relation "$\sim$" as follows: $a \sim b$ iff $a \cap b^{\perp}$, $a^{\perp} \cap b \in I_o$, $\bar{a} = \{b \in M: b \sim a\}$, $a \in M$, and $M/I_o = \{\bar{a}: a \in M\}$ is a Boolean σ-algebra [2], if we put $\bar{a}^{\perp} = \overline{a^{\perp}}$, and $\bigvee_i \bar{a}_i = \overline{\bigcup_i a_i}$.

Moreover, if m is a state, then $\mu(\bar{a}) := m(a)$, $a \in M$, is a probability measure on M/I_o.

Define a mapping $\bar{\tau}$: $M/I_o \longrightarrow M/I_o$ as follows: $\bar{\tau}(\bar{a}) = \overline{\tau a}$, $a \in M$. Then due to the invariancy of τ in m, $\bar{\tau}$ is a well-define homomorphism of M/I_o, that is,

(i) $\bar{\tau}(\bar{0}) = \bar{0}$;

(ii) $\bar{\tau}(\bar{a}^{\perp}) = (\bar{\tau}(\bar{a}))^{\perp}$, $a \in M$;

(iii) $\bar{\tau}(\bigvee_{i=1}^{\infty} \bar{a}_i) = \bigvee_{i=1}^{\infty} \bar{\tau}(\bar{a}_i)$, $\{a_i\} \subset M$.

Moreover, it is invariant in $\bar{\mu}$, i.e., $\bar{\mu}(\bar{\tau}(\bar{a})) = \bar{\mu}(\bar{a})$, $a \in M$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of F-observables of (X,M). Then $y_n: E \mapsto \bar{x}_n(E)$, $E \in B(R^1)$, $n \geq 1$, is an F-observable of M/I_o, i.e., it fulfils the same conditions as F-observables, moreover, $y_n(\emptyset) = \bar{0}$.

LEMMA 1. Let A be the minimal Boolean sub-σ-algebra of M/I_o containing all ranges of $\bar{\tau}^i \circ x_n$ for n = 1, 2, ... , i = = 1, 2, ... Then $\bar{\tau}A \subset A$.

PROOF. Denote by $A_o = \{\bar{a} \in A: \bar{\tau}\bar{a} \in A\}$. Then $\bar{0}, \bar{1} \in A_o$ and A is a Boolean sub-σ-algebra of M/I_o containing all ranges $\bar{\tau}^i \circ \bar{x}_i$, $n \geq 1$. Hence, $A_o = A$.

Q.E.D.

THEOREM 1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of F-observables of a fuzzy quantum space (X,M). Let $\{x_n\}_{n=1}^{\infty} \to \sigma$ a.e. [m], $\|x_n\| \leq K$, for n = 1, 2, ... Then

$$\frac{1}{n}\sum_{i=1}^{n} \bar{\tau}^i \circ x_i \to \sigma \text{ a.e. } [m].$$

PROOF. The Boolean sub-σ-algebra A in Lemma 1 has a countable genarator, hence, due to Varadarajan [8], there is an observable $z: B(R^1) \to M/I_o$, such that $A = z(B(R^1))$. It is clear that $\bar{\tau}$ is z-measurable, i.e., $\bar{\tau} \circ z(B(R^1)) \subseteq z(B(R^1))$. This is possible ([1]) iff there is a Borel measurable transformation $T: R^1 \to R^1$ such that $\bar{\tau}(z(E)) = z(T^{-1}(E))$, $E \in B(R^1)$. Therefore, $\bar{\tau}^k(z(E)) = z(T^{-k}(E))$, $E \in B(R^1)$, and due to Varadarajan [8], there exists a sequence of Borel measurable functions $\{f_n\}$: $\bar{x}_n(E) = z(f_n^{-1}(E))$ for every $E \in B(R^1)$, $n \geq 1$ and $\bar{\tau}^k \bar{x}_n(E) = \bar{\tau}^k(z\, f_n^{-1}(E)) = z(T^{-k}(f_n^{-1}(E)))$.

Moreover,

$\sigma(x_n) \supseteq \sigma(\bar{x}_n) \supseteq \sigma(f_n^{-1})$, then

$|f_n| \leq \|\bar{x}_n\| \leq \|x_n\| \leq K$.

It is clear, that μ_z: $E \mapsto \mu(z(E))$, $E \in B(R^1)$, is a probability measure on $B(R^1)$ and $(R^1, B(R^1), \mu_z, T)$ is a dynamic system. From a definition of the almost everywhere convergence $x_n \longrightarrow \mathcal{O}$ a.e. We have

$$m\left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} x_n([-\varepsilon,\varepsilon])\right) = 1 \iff$$

$$\mu\left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \bar{x}_n([-\varepsilon,\varepsilon])\right) = 1,$$

$$\mu\left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} z(f_n^{-1}([-\varepsilon,\varepsilon]))\right) = 1,$$

$$\mu_z\left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} f_n^{-1}([-\varepsilon,\varepsilon])\right) = 1 \text{ iff } f_n \longrightarrow 0 \quad \text{a.e. } [\mu_z].$$

Due to [6], we have

$$\frac{1}{n}\sum_{i=1}^{n} f_i \circ T^i \longrightarrow 0 \quad \text{a.e. } [\mu_z], \text{ so that}$$

$$\frac{1}{n}\sum_{i=}^{n} \tau^i \circ x_i \longrightarrow \mathcal{O} \quad \text{a.e. } [m].$$

Q.E.D.

Let x and y be two F-observables of (X,M). We say that $x \leqslant y$ if $\sigma(y - x) \subseteq [0,\infty)$.

THEOREM 2. Let $x_n \longrightarrow \mathcal{O}$ a.e. [m], $\mathcal{O} \leqslant x_n \leqslant y$ for n = 1, 2, 3, ... Then

$$\frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x_i \longrightarrow \mathcal{O} \quad \text{a.e. } [m].$$

PROOF. We used the similar arguments as those developed in Lemma 1 and Theorem 1. Let A_1 be the minimal Boolean sub-σ-algebra of M/I_0 containing all ranges of $\bar{\tau}^k \circ \bar{x}_n$ and $\bar{\tau}^k \circ \bar{y}$, for $k \geqslant 1$, $n \geqslant 1$. Then $\bar{\tau}A_1 \subseteq A_1$ and A has a countable generator. In view of Varadarajan [8], there is an observable z from $B(R^1)$ onto A_1. Moreover, there are f, f_n, $n \geqslant 1$, Borel measurable, real-valued functions, such that $\bar{x}_n(E) = z(f_n^{-1}(E))$, $\bar{y}(E) = z(f^{-1}(E))$, for any $E \in B(R^1)$, $n \geqslant 1$.

Denote $g_n = \max(0,f_n)$ and $g = \max(0,f)$, then we have $\bar{x}_n(E) = z(g_n^{-1}(E))$, $\bar{y}(E) = z(g^{-1}(E))$, $E \in B(R^1)$, $n \geqslant 1$.

Let $h_n = \min(f_n, f)$ and $h = f$, then $\bar{\bar{x}}_n(E) = z(h_n^{-1}(E))$, $\bar{\bar{y}}(E) = z(h^{-1}(E))$.

Moreover,

$0 \leqslant h_n \leqslant h$, and

$0 \leqslant \| \bar{\bar{x}}_n \| \leqslant \| \bar{\bar{y}} \|$.

Similarly as in the proof of Theorem 1, we obtain a dynamic system $(R^1, B(R^1), \mu_z, T)$, where $\mu_z(E) = \mu(z(E))$, $\mu_z : E \to \mu(z(E))$ is a probability measure on $B(R^1)$ and $\bar{\tau}(z(E)) = z(T^{-1}(E))$, $E \in B(R^1)$. This implies that

$$\frac{1}{n} \sum_{i=1}^{n} \tau^i \circ x_i \to \sigma \text{ a.e. } [m] \text{ iff}$$

$$\frac{1}{n} \sum_{i=1}^{n} \bar{\tau}^i \circ \bar{\bar{x}}_i \to \sigma \text{ a.e. } [\mu] \text{ iff}$$

$$\frac{1}{n} \sum_{i=1}^{n} h_i \circ T^i \to 0 \text{ a.e. } [\mu_z].$$

But the last assertion follows from the Theorem 2 of [6].

Q.E.D.

If x is an F-observable of (X, M) and $f(t) = |t|$, $t \in R^1$, then we put $|x| = f \circ x$.

COROLLARY 1. Let $x_n \to \sigma$ a.e., $|x_n| \leqslant y$ for $n = 1, 2, \ldots$. Then

$$\frac{1}{n} \sum_{i=1}^{n} \tau^i \circ x_i \to \sigma \text{ a.e. } [m].$$

PROOF. We denote $x_n = x_n^+ - x_n^-$, where $x_n^+ = f^+ \circ x_n$, $x_n^- = f^- \circ x_n$, $f^+(t) = \max(0,t)$ and $f^-(t) = -\min(0,t)$, $|x| = x^+ + x^-$. Applying Theorem 2 to both $\{x_n^+\}$ and $\{x_n^-\}$ we get what was claimed.

Q.E.D.

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