A NOTE TO STRONG LAW OF LARGE NUMBERS FOR RANDOM VARIABLES WITH VALUES IN THE FUZZY REAL LINE

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Introduction

Klement in [3] introduced random variables with values in the fuzzy real line and proved a Strong Law of Large Numbers. We generalize this result for pairwise independent, identically distributed fuzzy-valued random variables. Finally we indicate some another classical results of the probability theory which might be introduced for the fuzzy-valued random variables, too.

Fuzzy real line and fuzzy-valued random variables

Here we recall the concept developed by Klement in [3]. Throughout this paper, R will denote the real line, and $\overline{R} = R \cup \{-\infty, +\infty\}$. For the unit interval $[0,1]$ we shall write I. The extended fuzzy real line $\overline{R}(I)$ is the set of all cumulative distribution functions on $\overline{R}$, i.e. the set of all functions $p:\overline{R} \longrightarrow I$ such that

$$p(-\infty) = 0 \ , \ p(+\infty) = 1 \ , \forall r \in R: \ p(r) = \sup\{p(s), \ s < r\} \qquad (1)$$

A natural interpretation of a fuzzy number p is the following: p(r) is the degree to which p is less than the (nonfuzzy) number r. Any nonfuzzy real number r is identified with $\delta_r = 1_{]r,+\infty]}$. A fuzzy number p is said to be finite if $\inf\{p(r), \ r \in R\} = 0$ and $\sup\{p(r), \ r \in R\} = 1$. The set of all finite fuzzy numbers will be denoted R(I). For more details see [3].

Consider closed intervals $[a,b]$ and $[c,d]$ on $\overline{R}$ and a left-continuous nondecreasing function $f:[a,b] \longrightarrow [c,d]$ with $f(a) = c$.

Then its quasi-inverse $f^q:[c,d]\longrightarrow[a,b]$ defined by

$$f^q(c) = a \ , \ f^q(s) = \sup\{r \in [a,b],\ f(r) < s\} \ \text{ for } s \in\ ]c,d] \quad (2)$$

is again left-continuous and nondecreasing, and $(f^q)^q = f$.
If $\bar{R}^q(I)$ denotes the set of all quasi-inverses of fuzzy numbers p in $\bar{R}(I)$ then q is an involution from $\bar{R}(I)$ onto $\bar{R}^q(I)$ and it is possible to introduce an algebraic structure on $\bar{R}(I)$ as follows:

$$p \le k \Longleftrightarrow [p]^q\,(t) \le [k]^q\,(t) \quad \text{for all } t \in I \quad (3)$$

$$[p \oplus k]^q\,(t) = [p]^q\,(t) + [k]^q\,(t) \quad (4)$$

$$[p \circledast k]^q\,(t) = \sup\{[p^+]^q\,(v).[k^+]^q\,(v) + [p^+]^q\,(1-v).[k^-]^q\,(v) + + [p^-]^q\,(v).[k^+]^q\,(1-v) + [p^-]^q\,(1-v).[k^-]^q\,(1-v)\ ;\ v < t\} \quad (5)$$

Here in the latter formula p^+ and p^- are fuzzy numbers as follows:

$$p^+(r) = p(r).1_{]0,+\infty]} \ , \ p^-(r) = p(r).1_{[-\infty,0]} + \delta_0 \quad (6)$$

It is clear that p is finite iff $[p]^q\,(t)$ is finite for any t , $t \in\ ]0,1[$. A sequence $(p_n)_{n\ N}$ in $\bar{R}(I)$ converges to p if for all continuity points of p we have

$$p(r) = \lim p_n(r) \quad (7)$$

Then $(p_n) \longrightarrow p$ iff $([p_n]^q) \longrightarrow [p]^q$.
Let $(\Omega, \mathcal{A}, P)$ be a probability space. A measurable function $X:\Omega \longrightarrow \bar{R}(I)$ will be called fuzzy-valued random variable. Its quasi-inverse $[X]^q:\Omega \longrightarrow \bar{R}^q(I)$ is defined by $X^q(\omega) = [X(\omega)]^q$ and it is measurable iff X is measurable. The expected value of X is given by

$$EX = [E(X^q)]^q \quad (8)$$

where $E(X^q)\,(t)$ is (the classical) expected value of the real valued random variable $[X(.)]^q\,(t)$. This extends the classical expected value of a real-valued random variable $X:\Omega \longrightarrow \bar{R}$ in the sense that $E(i(X)) = \delta_{EX}$, i beeing the embedding from $\bar{R}$ into $\bar{R}(I)$. It is easy to see that introduced expected value of fuzzy-valued random variables is a linear functional.

Strong Law of Large Numbers

Theorem 1 (Klement, 3). Let X_1, X_2,... be a sequence of independent, identically distributed random variables with values in R(I) and $E(X_1^2)$ finite. Then

$$\frac{1}{n} \odot (X_1 \oplus X_2 \oplus \dots \oplus X_n) \longrightarrow E(X_1) .$$

Etemadi in[2] and Deniel in [1] have proved a Strong Law of Large Numbers for pairwise independent, identically distributed real-valued random variables. Etemadi's elementary proof deals with the finiteness of $E|X_1|$. Since it is not clear what the absolute value of a fuzzy number, i.e. of a fuzzy -valued random variable should be, we improve the finiteness condition upon X^+ and X^- variables, see [4] .

Theorem 2. Let X_1, X_2,... be a sequence of pairwise independent, identically distributed random variables with values in R(I) and let $E(X_1^+)$ and $E(X_1^-)$ be finite. Then

$$\frac{1}{n} \odot (X_1 \oplus X_2 \oplus \dots \oplus X_n) \longrightarrow E(X_1) .$$

Proof. For each $t \in [0,1]$ the real-valued random variables $[X_1(.)]^q (t)$, $[X_2(.)]^q (t)$,... are pairwise independent and identically distributed (see e.g. [3]). For any fuzzy number $p \in R(I)$ we have $[p^+]^q (t) = ([p]^q (t))^+$ and $[p^-]^q (t) = -([p]^q (t))^-$ (recall that for a real number r we define $r^+ = \max\{0,r\}$ and $r^- = \max\{0,-r\}$). The finiteness of the fuzzy numbers $E(X_1^+)$ and $E(X_1^-)$ implies the finiteness of $[E(X_1^+)]^q (t)$ and $[E(X_1^-)]^q (t)$ and consequently the finiteness of $E|[X_1]^q(t)|$ for any $t \in]0,1[$. Then by Etemadi [2] we get for any $t \in]0,1[$:

$$\frac{1}{n} \odot [(X_1 \oplus X_2 \oplus \dots \oplus X_n)]^q(t) = \frac{1}{n} \circ (X_1^q + X_2^q + \dots + X_n^q)(t) \longrightarrow$$

$\longrightarrow E(X_1^q)(t) = [E(X_1)]^q (t)$, what makes our proof complete.

Remarks

The key problem of investigation of another properties of random variables with values in the fuzzy real line is the problem of measurability. Then we suggest to study another analogies of the probability theory. We give some examples of possible directions of the next investigation:

i) Birkhoff's Individual Ergodic Theorem (for a given measure preserving transformation φ)

ii) conditional expectation, martingales, martingale convergence theorems

iii) Radon-Nikodem Theorem.

References

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