ON T-SUM OF FUZZY NUMBERS*

R.Fullér, Computer Center, L.Eötvös University,
H-1502 Budapest 112, pf.157

We study the problem: if $\tilde{a}_i$ $i \in \mathbb{N}$ are fuzzy numbers of triangular or exponential form, what is the membership function of the infinite sum $\tilde{a}_1+\tilde{a}_2+\ldots$ (defined via a sup- t-norm convulution)?

Keywords: t-norm, extension principle, fuzzy variable

DEFINITIONS.

A fuzzy number is a fuzzy set on the real line with a unimodal, upper semicontinuous and normalized membership function.

A function $T:[0,1]\times[0,1]\to[0,1]$ is said to to be triangular norm (t-norm for short)iff T is commutative associative, non-decreasing and $T(x,1)=x$, $x\in[0,1]$.

We shall use the following t-norms: $\min\{x,y\}$, xy, $\max\{x+y-1,0\}$ and $T_W(x,y)$, where $T_W(x,y)=x$ if $y=1$, $T_W(x,y)=y$ if $x=1$ and $T_W(x,y)=0$ otherwise.

Let T be a t-norm. If $\tilde{a}$ and $\tilde{b}$ are fuzzy numbers, then the membership function of their T-sum $\tilde{a}+\tilde{b}$ is defined as [1]

$$(\tilde{a}+\tilde{b})(z) = \sup_{x+y=z} T(\tilde{a}(x),\tilde{b}(y)).$$

....

*This work has been supported by Young Scholars' Fund under No.400-0113.

A symmetric triangular fuzzy number $\tilde{a}$ denoted by (a,α) is defined as $\tilde{a}(t)=1 - |a-t|/\alpha$ if $|a-t|\leq\alpha$ and $\tilde{a}(t)=0$ if $|a-t|>\alpha$, where $\alpha>0$ is the width and $a\in\mathbf{R}$ is the centre of $\tilde{a}$.

A symmetric exponential fuzzy number $\tilde{a}$ denoted by $(a,\alpha)_e$ is defined as $\tilde{a}(t) = \exp(-|a-t|/\alpha)$, $\alpha>0$.

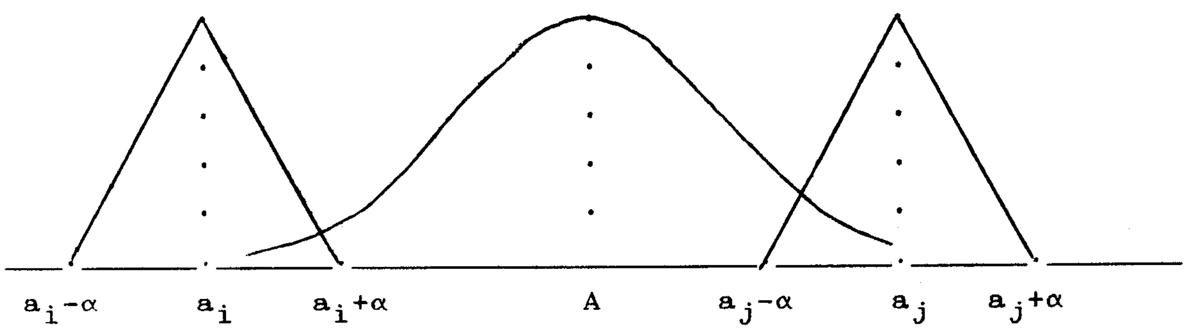
THEOREMS.

Theorem 1. Let $T(x,y)=xy$ and $\tilde{a}_i = (a_i,\alpha)$, $i \in \mathbb{N}$. If $A:=\sum_{i=1}^{\infty} a_i$ exists and it is finite, then with the notation

$$\tilde{A}_n := \tilde{a}_1+\ldots+\tilde{a}_n, \ n\in\mathbb{N},$$

we have

$$(\lim_{n\to\infty} \tilde{A}_n)(z) = \exp(-|A-z|/\alpha), \ z\in\mathbb{R}.$$



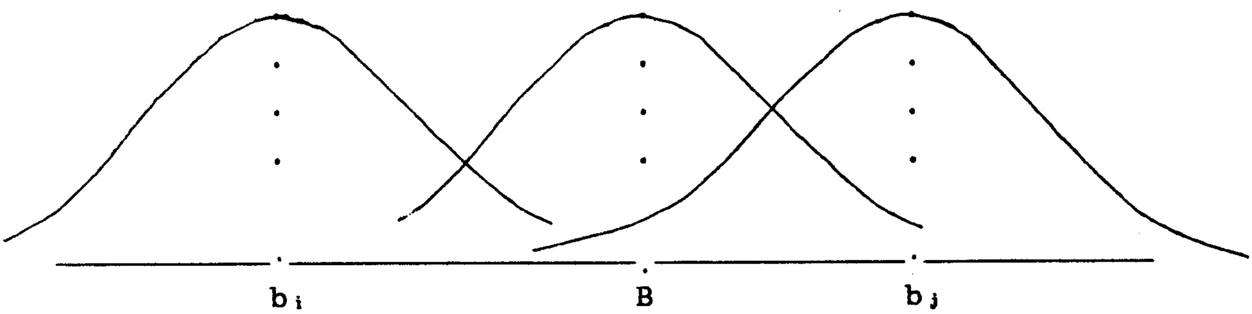
Remark 1. This theorem can be interpreted as a *central limit theorem for mutually T-related fuzzy variables* $\tilde{a}_i, i\in\mathbb{N}$ (see [3]).

Theorem 2. Let $T(x,y)=xy$ and $\tilde{b}_i = (b_i,\beta)_e$, $i\in\mathbb{N}$ .If $B:=\sum_{i=1}^{\infty} b_i$ exists and it is finite, then with the notation

$$\tilde{B}_n := \tilde{b}_1+\dots+\tilde{b}_n, \quad n\in\mathbb{N},$$

we have

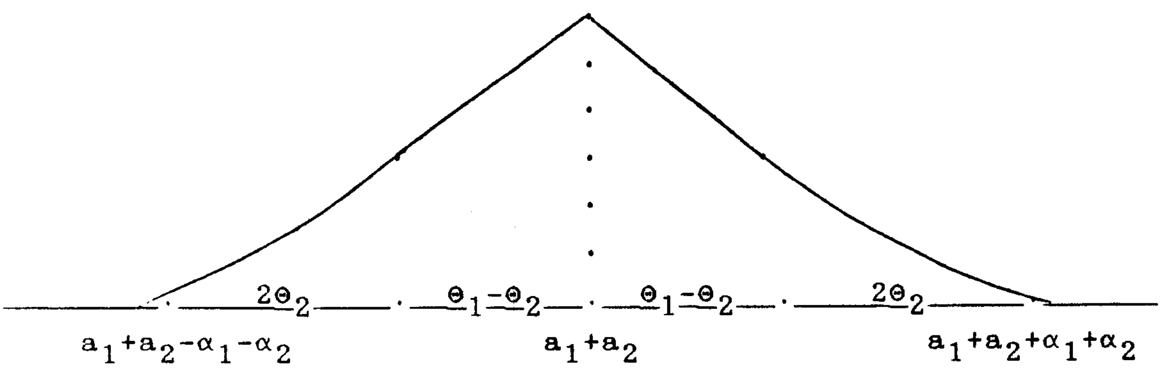
$$(\lim_{n\to\infty} \tilde{B}_n)(z) = \exp\left(-|B-z|/\beta\right), \quad z\in\mathbb{R}.$$



Theorem 3. Let $T(x,y)=xy$ and $\tilde{a}_i=(a_i,\alpha)$, $i=1,2$. Then

$$(\tilde{a}_1+\tilde{a}_2)(z)=\begin{cases} 1-\dfrac{|a_1+a_2-z|}{\alpha_1\vee\alpha_2} & \text{if } |a_1+a_2-z|\leqq|\alpha_1-\alpha_2|, \\[2ex] \dfrac{(\alpha_1+\alpha_2)^2}{4\alpha_1\alpha_2}\left(1-\dfrac{|a_1+a_2-z|}{\alpha_1+\alpha_2}\right)^2 & \text{otherwise}, \\[2ex] 0 & \text{if } |a_1+a_2-z|>\alpha_1+\alpha_2, \end{cases}$$

where $\alpha_1\vee\alpha_2 = \max\{\alpha_1,\alpha_2\}$. Let $\Theta_1=\alpha_1\vee\alpha_2$, $\Theta_2=\alpha_1\wedge\alpha_2$.



"Product"-sum of two triangular fuzzy numbers

Theorem 4. Let $T(x,y)=xy$ and $\tilde{a}_i=(a_i,\alpha_i)$, and let θ_i be the i-th greatest width, $i=1,\ldots,n$. Then with the notations

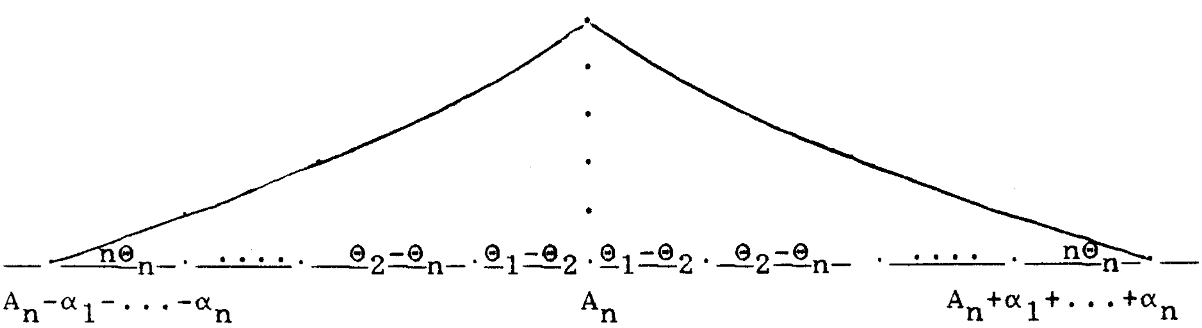
$$\tilde{A}_n:=\tilde{a}_1+\ldots+\tilde{a}_n,\quad A_n:=a_1+\ldots+a_n$$

we have

$$\tilde{A}_n(z)=\begin{cases} 1-\dfrac{|A_n-z|}{\theta_1} & \text{if } z\in M_1\\ \vdots & \vdots\\ \dfrac{(\theta_1+\ldots\theta_k)^k}{k^k\prod_{i=1}^{k}\theta_i}\left[1-\dfrac{|A_n-z|}{\theta_1+\ldots+\theta_k}\right]^k & \text{if } z\in M_k\\ \vdots & \vdots\\ \dfrac{(\theta_1+\ldots+\theta_n)^n}{n^n\prod_{i=1}^{n}\theta_i}\cdot\left[1-\dfrac{|A_n-z|}{\theta_1+\ldots+\theta_n}\right]^n & \text{if } z\in M_n \end{cases}$$

where

$$M_1=\{z\in\mathbb{R}:|A_n-z|\leqq\theta_1-\theta_2\},$$
$$M_2=\{z\in\mathbb{R}:\theta_1-\theta_2\leqq|A_n-z|\leqq\theta_1+\theta_2-2\theta_n\},$$
$$\ldots$$
$$M_k=\{z\in\mathbb{R}:\theta_1+\ldots+\theta_{k-1}-(k-1)\theta_n\leqq|A_n-z|\leqq\theta_1+\ldots\theta_k-k\theta_n\},$$
$$\ldots$$
$$M_n=\{z\in\mathbb{R}:\theta_1+\ldots+\theta_{n-1}-(n-1)\theta_n\leqq|A_n-z|\leqq\theta_1+\ldots+\theta_n\},$$



"Product"-sum of n triangular fuzzy numbers.

Remark 2. This theorem shows that if each $\tilde{a}_i$ has the same width $\alpha>0$, then the membership function of $\tilde{A}_n$ is the following

$$\tilde{A}_n(z)=\begin{cases}\left[1-\dfrac{|A_n-z|}{n\alpha}\right]^n & \text{if } |A_n-z| \leqq n\alpha,\\ 0 & \text{otherwise,}\end{cases}$$

Theorem 5. Let $T(x,y)=T_W(x,y)$ and $\tilde{a}_i=(a_i,\alpha_i)$, $1\leqq i\leqq n$. Then with the notations $\tilde{A}_n=\tilde{a}_1+\ldots+\tilde{a}_n$ and $A_n=a_1+\ldots+a_n$ we have

$$\tilde{A}_n(z)=\begin{cases}1-\dfrac{|A_n-z|}{\max \alpha_i} & \text{if } |A_n-z| \leqq \max \alpha_i,\\ 0 & \text{otherwise.}\end{cases} \qquad (1)$$

Theorem 6 [4]. Let $T(x,y)=\max\{x+y-1,0\}$ and $\tilde{a}_i=(a_i,\alpha_i)$, $1\leqq i\leqq n$. Then the fuzzy number $\tilde{a}_1+\ldots+\tilde{a}_n$ has the membership function of (1).

Concluding remarks. Similar results can be obtained for the t-sum of non-symmetrical fuzzy numbers of triangular (or exponential) form by the help of their decomposition into non-decreasing and non-increasing parts (see [2]).

QUESTION.

Let $T(x,y)=xy$ and let $\tilde{a}_i = (a_i,\alpha,\beta)$, $1\leqq i\leqq n$ be fuzzy numbers of LR-type. On what condition will the membership function of the "product"-sum $\tilde{A}_n:=\tilde{a}_1+\ldots\tilde{a}_n$ have the following form

$$\tilde{A}_n(z)=\begin{cases} L^n\left(\frac{A_n-z}{n\alpha}\right) & \text{if } z \leqq A_n, \\ R^n\left(\frac{z-A_n}{n\beta}\right) & \text{if } z \geqq A_n. \end{cases}$$

REFERENCES.

[1] Dubois D. and Prade H.,Additions of Interactive Fuzzy Numbers,IEEE Transactions on Automatic Control, 1981,Vol.26, 926-936.

[2] Dubois D. and Prade H.,Inverse Operations for Fuzzy Numbers, Proc. IFAC Symp. on Fuzzy Information, Knowledge,Representation and Decision Analysis, Marseille,1983,399-404.

[3] Rao,M.B.and Rashed A.,Some comments on fuzzy variables, Fuzzy Sets and Systems, 6(1981) 285-292.

[4] Kovács M., On g-fuzzy linear systems, BUSEFAL,1989,No.37, 69-78.