FUZZY SINGULAR MATRIX

Yi Chong-Xin
Dept. of Basis. Qiqihar College of Light Industry
Qiqihar, CHINA

Tang Xiang
Dept. of Basis. Qiqihar College of Light Industry
Qiqihar, CHINA

ABSTRACT

This paper put forward the concept of fuzzy singular matrix and has a elementary discussion on its propertys. And also gives two theorems by which the nonsingularity of fuzzy matrix can be discriminated. It is a supplement to one in the reference [1].

Keyword : Fuzzy singular matrix

1. BASIC CONCEPTS

Let's discuss the problems in semisimple ring I = ([0,1], a+b = max{a,b}, a×b = min{a,b}, for any a,b∈[0,1])

Definition 1.1. For any two fuzzy matrixes $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$, A plus B is matrix $(a_{ij}+b_{ij})_{m\times n}$, i.e.

$$A + B = (a_{ij}+b_{ij})_{m\times n}$$

Definition 1.2. For any a scalar $K\in[0,1]$ and any a fuzzy matrix $A = (a_{ij})_{m\times n}$, K times A equals $(K\cdot a_{ij})_{m\times n}$, i.e.

$$K\cdot A = (K\cdot a_{ij})_{m\times n}$$

Definition 1.3. For any two fuzzy matrix $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times t}$, A multiplied by B equals matrix

$$(\sum_{\lambda=1}^{n} a_{i\lambda}b_{\lambda j})_{m\times t}, \quad \text{i.e.} \quad AB = (\sum_{\lambda=1}^{n} a_{i\lambda}b_{\lambda j})_{m\times t}.$$

Definition 1.4. Let H be the set of all 1×n order fuzzy matrixes (fuzzy row vectors with n elements) and $A_1 \cdots A_r$, $A\in H$, if there are r numbers $a_1 \cdots a_r \in[0,1]$ which will make the formula

$$A = \sum_{i=1}^{r} a_i A_i$$

hold true, A is said to be a linear combination of the row matrixes (row vectors) $A_1 \cdots A_r$. Further, suppose there be two grops of fuzzy row vectors $A_1 \cdots A_m \in H$ and $B_1 \cdots B_k \in H$, and let A be a vector set consisted of row matrixes $A_1 \cdots A_m$ and B be an other vector set consisted of row matrixes $B_1 \cdots B_k$. The vector set A is said to be a linear combination of the vector set B or one can say that A may be linearly expressed through B, if vectors of the A are all expressed as linear combinations of the row matrixes $B_1 \cdots B_k$.

Definition 1.5. For any a m×n order fuzzy matrix A, the set of linear combinations of all row vectors of the A is called the row space of A and expressed as R(A).

Definition 1.6. Let A be a m×n order fuzzy matrix and G be a set of the some vectors of the R(A), if the R(A) may be linearly expressed through G, call the G a generating set of the R(A). Further, if removing any a vector from the G, the set being consisted of the remainers of the G is no longer generating set of the R(A), G is called a minimal generating set of the R(A).

Definition 1.7, For any a m×n order fuzzy matrix A. row rank of A is the vector numbers of minimal generating set generating R(A), and expressed as $\rho_r(A)$. Similarily, we may define column space C(A) and column rank $\rho_c(A)$ of matrix A.

Definition 1.8. For any m×n order fuzzy matrix A, we say that A is having rank if $\rho_r(A) = \rho_c(A) = r$, and then the number r is called rank of matrix A and expressed as $\rho(A)$.

Definition 1.9. For any m×n order fuzzy matrix A, Schein rank of A is the minimum number of the matrixes which have rank 1 and are then added together making A.

Definition 1.10. For any two fuzzy non-vornishing vectors $X = (x_1, x_2 \cdots x_m)$ and $Y = (y_1, y_2 \cdots y_n)$, the fuzzy matrix $(x_i \cdot y_j)_{m\times n}$ is called a cross product multiplied X by Y and expressed as (X,Y), i.e.

$$(X,Y) = X^T \cdot Y = (x_i \cdot y_j)_{m\times n}$$

Thereby, K. H. Kim and Roush [6] pointed out the schein rank $\rho_s(A)$ of fuzzy matrix A is namely the minimum number of cross products, the sum of which is just the A.

2. FUZZY SINGULAR MATRIX

In the reference [1], a fuzzy non-singular matrix is defined as follows :

A m×n order fuzzy matrix A is said to be non-singular if $\rho_r(A) = m$ and $\rho_c(A) = n$.

Here we give another definition which is fully contrary to the non-singular one in concept.

Definition 2.1. For any m×n order fuzzy matrix A, A is called fuzzy singular matrix if A is not non-singular.

For example, let

$$A = \begin{bmatrix} 0.9 & 0.8 \\ 0.8 & 0.7 \\ 0.6 & 0.5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0.8 & 0.4 \\ 0.6 & 0.3 \\ 0.4 & 0.2 \end{bmatrix}$$

then the matrix A is non-singular, because $\rho_r(A)$ is 3 (row numbers of A) and $\rho_c(A) = 2$ (column numbers of A). whereas B is singular, because $\rho_c(B) = 1$ and $\rho_c(B) < 2$ (column numbers of the B).

The reference [1] gives dicision theorems of fuzzy non-singular matrix as follows :

A m×n order fuzzy matrix A is non-singular if and only if :

Whole row and column of the A are all linear independent ; $\rho_r(A) = \rho_c(A) = \rho(A) = n$; $\rho_s(A) = n$.

As a futher supplement to the reference (1), we give following two decision theorems.

Theorem 2.1. A n×n order fuzzy matrix A is non-singular if and only if fuzzy relational non-deterministic equation of the A :

$$A = Y_{n\times t}X_{t\times n} \qquad (2.1)$$

has not any solution as exponent t = n-1.

Proof. If equation (2.1) has no solution as exponent t = n-1, we may pronounce that equation (2.1) will have the solution as exponent t = n. Because

$$Y_{n\times n} = A_{n\times n} \qquad \text{and}$$

$$X_{nn} = \begin{bmatrix} \max\limits_i\{a_{i1}\} & 0 & \cdots & 0 \\ 0 & \max\limits_i\{a_{i2}\} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \max\limits_i\{a_{in}\} \end{bmatrix}_{n\times n}$$

is namely a set of solutions of the equation (2.1).

On the basis of the theorem 1.2 of the reference (2) and the theorem 3.1 of the reference (3), we may know schein rank of the A is equal to n, i.e. $\rho_s(A) = n$, therefore the A is non-singular.

On the contrary, if the A is non-singular, $\rho_s(A) = n$. In accordence with the theorem 3.1 of the reference (3), the fuzzy relational non-deterministic equation of the A (2.1) has no solution as exponent t = n-1.

Thereby, we may abtain the following theorem on the moment.

Theorem 2.2 A m×n order fuzzy matrix A is singular if and only if the fuzzy relational non-deterministic equation of A (2.1) has just a solution as exponet t = n-1.

Theorem 2.3. If a n×n order fuzzy matrix A can be expressed as sum of n cross products and the cross product numbers can not be reduced again, the A is non-singular.

In fact, as the cross products are all like this fuzzy matrix whose rank is one and the A can be expressed as sum of n cross products and the numbers of the cross number can not be reduced again, $\rho_s(A) = n$. Therefore the A is a non-singular fuzzy matrix.

Theorem 2.4. $(\dot{1})$ If $A_{m\times n}$ is a singular fuzzy matrix and $B_{n\times k}$ is a non-singular fuzzy matrix, AB is still a singular fuzzy matrix.

$(\dot{2})$. If $A_{m\times n}$ is a singular fuzzy matrix and so is $B_{n\times k}$, AB is a singular fuzzy matrix yet.

$(\dot{3})$. If $A_{m\times n}$ is a non-singular fuzzy matrix and $B_{n\times k}$ is a singular fuzzy matrix, AB is a singular fuzzy matrix.

Proof. $(\dot{1})$ Because $A_{m\times n}$ is a fuzzy singular matrix, according to theorem 2.2, we may let :

$$A = Y_{m\times(n-1)}X_{(n-1)\times n}$$

The Y and X here are all fuzzy matrix. Whence, we have formula like this :

$$\begin{aligned} AB &= (Y_{m\times(n-1)}X_{(n-1)\times n})B_{n\times k} \\ &= Y_{m\times(n-1)}(X_{(n-1)\times n}B_{n\times k})_{(n-1)\times k} \end{aligned}$$

It means that the fuzzy relational non-deterministic equation of the AB has a set of solution at less as $t = n-1$. In fact, the $Y_{n\times(n-1)}$ and $(XB)_{(n-1)\times k}$ is namely one of them. Thus we may know that the matrix AB is a singular matrix in accordence with the theorem 2.2.

On the basis of these same reasons, we may prove that conclution $(\dot{2})$ and $(\dot{3})$ are all true.

Let A be a nonzero fuzzy row vector (i.e. $A = (a_{11}, a_{12}, \ldots$

a_{1m})) with m elements ($m>1$). Distingctly $\rho_r(A) = 1$ and $\rho_c(A)<m$ (column numbers). Therefore A is a singular fuzzy matrix. Because of the same reason, a nonzero fuzzy column vector is said to be a singular fuzzy matrix if and only if its elements are more than one. Thus the theorem has become like this :

Theorem 2.5. Any non-zero fuzzy vectors whose element numbers are more than one are all singular fuzzy matrixes.

Theorem 2.6. Suppose A is a mxn order fuzzy matrix if the minimum elements of the A form some rows (or columns) of the A, the A is a singular matrix.

Proof. For conveninnce, we may suppose that the minimum elements form the first row of the A. Thus we may let

$$A = \begin{bmatrix} a_{11} & a_{11} & \cdots & a_{11} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

in which $a_{11} \leq \min_{i,j}\{a_{i,j}\}$. Whence, the equal formular
$(a_{11}, a_{11}, \cdots a_{11}) = a_{11}(a_{21} \cdots a_{2n}) + 0(a_{31} \cdots a_{3n}) + \cdots + 0(a_{m1} \cdots a_{mn})$
hold true. That is : the first row may be expressed as linear combination of others of the A. From this, we may know the unequal formula

$$\rho_r(A) \leq m-1$$

is true. Therefore the A is singular. Q.E.D.

The reference(5) introduces a concept of "successive elimination". From this, we have following definition, too.

If the mininum elements of the A form some rows (or columns) with same elements, the rows (or columns) are called eliminated ones. And after eliminating them, the fuzzy matrix consisting of the remainders of the A in its original sequence is called fuzzy submatrix of the A.

For example, let

$$A = \begin{bmatrix} 0.7 & 0.8 & 0.2 & 0.4 & 0.5 \\ 0.3 & 0.3 & 0.2 & 0.3 & 0.3 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.7 & 0.9 & 0.2 & 0.4 & 0.5 \\ 0.6 & 0.6 & 0.2 & 0.4 & 0.5 \end{bmatrix}$$

Now we write off the rows and columns concerning of the A step by step and can get the result like this :

$$\left[\begin{array}{ccccc} 0.7 & 0.8 & 0.2 & 0.4 & 0.5 \\ 0.3\cdots & 0.3\cdots & 0.2\cdots & 0.3\cdots & 0.3\cdots \\ 0.1\cdots & 0.1\cdots & 0.1\cdots & 0.1\cdots & 0.1\cdots \\ 0.7 & 0.9 & 0.2 & 0.4 & 0.5 \\ 0.6\cdots & 0.6\cdots & 0.2\cdots & 0.4\cdots & 0.5\cdots \end{array}\right]\begin{array}{c} \\ \cdots 3 \\ \cdots 1 \\ \\ \cdots 6 \end{array} \qquad (2.2)$$

$$\begin{array}{ccccc} \vdots & & \vdots & \vdots & \vdots \\ 7 & & 2 & 4 & 5 \end{array}$$

In the formula (2.2), the rows (or columns) connected by a dotted line express ones writen off and the numbers marked out of the A indicate the sequence of successive elimination. Through this eliminative process, the A becomes a submatrix

$$A_1 = \begin{bmatrix} 0.8 \\ 0.9 \end{bmatrix} \qquad \text{finaly.}$$

Thus the theorem 2.6 can also be expressed like this :

Theorem 2.7. For any m×n order fuzzy matrix A, if it can be eliminated one row (or one column) at least through "successive elimination", the A is a fuzzy singular matrix.

For example, the fuzzy matrix A appering in the preceding example is namely a singular one.

Inference for any m×n order fuzzy matrix A only if the elements of its some row (or column) are all zero, the A is a fuzzy singular matrix.

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