GREY TOPOLOGICAL SPACE
(COMPOSITION FUZZY T.S.)

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ABSTRACT: In this paper the definitions and their concepts of grey topological space are introduced. And on this basis, the relationship between grey topological spaces and fuzzy and general topological space is discussed. There are also studies of grey continuous mapping and compact grey topological space.

KEYWORDS: Grey subset, grey point, grey topological space and grey continuous mapping.

I. INTRODUCTION

C.L.Chang defined fuzzy topological space in 1968. Mr Wu Heqin and Wang Qingyin gave the conception and properties of grey sets. We shall study the grey topological space on this basis.

Definition 1: Let X be a discussible field. If upper and lower subordinate functions of grey subset A in X are all equal to 1, or $\bar{\mu}_A(x)=\underline{\mu}_A(x)=1, \forall x \in X$, then A is called the whole grey set in X, usually written X.

If upper and lower subordinate functions of grey subset A in X are all equal to 0, or $\bar{\mu}_A(x)=\underline{\mu}_A(x)=0, \forall x \in X$. Then A is called empty grey set in X, usually written $\emptyset$.

Definition 2: Let X be a discussible field. The grey subset in X

$$\bar{\mu}(x)=\begin{cases}\bar{\lambda}, & x=a\\ 0, & x\neq a\end{cases} \qquad \text{where } \bar{\lambda}\neq 0,$$

$$\underline{\mu}(x)=\begin{cases}\lambda, & x=a\\ 0, & x\neq a\end{cases} \qquad \text{where } \lambda\neq 0.$$

These are called the grey point of x, written $a\bar{\lambda}\lambda$ or simplified $a\bar{\lambda}\lambda$ to a.

Definition 3: Let A, B be grey subsets of X. If $\min[\underline{\mu}_A(a), \underline{\mu}_B(a)]>0$, then A and B are called the joint to point a.

Definition 4: Let A, B be grey subsets of X. If $\underline{\mu}_A(a)+\underline{\mu}_B(a)>1$, then A and B coincide in point a. A and B are called coincide if and only if there exists $a\in X$ such that A and B coincide to a.

Definition 5: Let A be a grey subset of X, $a\bar{\lambda}\lambda$ be a grey point of X.

(1) If $0<\lambda\leq\bar{\lambda}\leq\underline{\mu}_A(a)$, then $a\bar{\lambda}\lambda$ is called belong to A, usually written $a\bar{\lambda}\lambda\in A$.

(2) If $\lambda+\underline{\mu}_A(a)>1$, then $a\bar{\lambda}\lambda$ is called coincide to A, usually written $a\bar{\lambda}\lambda\triangle A$.

Definition 6: Let f be a mapping from set X to Y, B is a grey subset in Y and subordinate functions of B are $\bar{\mu}_B(y)$, $\underline{\mu}_B(y)$, $y\in Y$. By the inverse image of B(written $f^{-1}[B]$) we mean grey subset in X and its subordinate functions are defined by:

$$\bar{\mu}_{f^{-1}[B]}(x)=\bar{\mu}_B(f(x)), \quad \forall x\in X,$$
$$\underline{\mu}_{f^{-1}[B]}(x)=\underline{\mu}_B(f(x)), \quad \forall x\in X.$$

Let A be a grey subset in X and subordinate functions of A are $\bar{\mu}_A(x)$, $\underline{\mu}_A(x)$, $x\in X$. By the image of A(written $f[A]$), we mean grey subset in Y and its subordinate functions are defined by: $\forall y\in Y$

$$\bar{\mu}_{f[A]}(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)}\{\bar{\mu}_A(x)\} & f^{-1}(y)\neq\emptyset\\ 0 & f^{-1}(y)=\emptyset\end{cases}$$

$$\underline{\mu}_{f[A]}(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)}\{\underline{\mu}_A(x)\} & f^{-1}(y)\neq\emptyset\\ 0 & f^{-1}(y)=\emptyset\end{cases}$$

where $f^{-1}(y)=\{x \mid f(x)=y\}$.

Theorem 1 Let f be a mapping from set X to Y, B is a grey subset in X, then (1) $f^{-1}[B^C]=(f^{-1}[B])^C$.

(2) If f is a surjetion, then $f[f^{-1}[B]]=B$.

(3) And g is a mapping from Y to set Z, then there exists any grey subset C of Z: $(g\circ f)^{-1}[C]=f^{-1}[g^{-1}[C]]$,

where $g\circ f$ is the compound mapping of f and g.

Proof: (1) Since $\bar{\mu}_{f^{-1}[B^c]}(x)=\bar{\mu}_{B^c}(f(x))=1-\underline{\mu}_B(f(x))$

$=1-\underline{\mu}_{f^{-1}[B]}(x)=\bar{\mu}_{(f^{-1}[B])^c}(x), \quad \forall x\in X.$

$\underline{\mu}_{f^{-1}[B^c]}(x)=\underline{\mu}_{B^c}(f(x))=1-\bar{\mu}_B(f(x))=1-\bar{\mu}_{f^{-1}[B]}(x)$

$=\underline{\mu}_{(f^{-1}[B])^c}(x), \quad \forall x\in X.$

Hence $f^{-1}(B^C)=(f^{-1}[B])^C$.

(2) Since f is a surjetion, hence $f^{-1}(y)\neq 0$.

Thus $\bar{\mu}_{f[f^{-1}[B]]}(y)=\sup\limits_{x\in f^{-1}(y)}\{\bar{\mu}_{f^{-1}[B]}(x)\}=\sup\limits_{x\in f^{-1}(y)}\{\bar{\mu}_B(f(x))\}=\bar{\mu}_B(y)$,

$\underline{\mu}_{f[f^{-1}[B]]}(y)=\sup\limits_{x\in f^{-1}(y)}\{\underline{\mu}_{f^{-1}[B]}(x)\}=\sup\limits_{x\in f^{-1}(y)}\{\underline{\mu}_B(f(x))\}=\underline{\mu}_B(y)$.

So $f[f^{-1}[B]]=B$.

(3) For $\forall x\in X$, $\bar{\mu}_{(g\circ f)^{-1}[C]}(x)=\bar{\mu}_C[(g\circ f)(x)]=\bar{\mu}_C[g[f(x)]]$

$=\bar{\mu}_{g^{-1}[C]}[f(x)]=\bar{\mu}_{f^{-1}[g^{-1}[C]]}(x)$.

$\underline{\mu}_{(g\circ f)^{-1}[C]}(x)=\underline{\mu}_C[(g\circ f)(x)]=\underline{\mu}_C[g[f(x)]]$

$=\underline{\mu}_{g^{-1}[C]}[f(x)]=\underline{\mu}_{f^{-1}[g^{-1}[C]]}(x)$.

Hence $(g\circ f)^{-1}[C]=f^{-1}[g^{-1}[C]]$.

II CONCEPTION OF GREY TOPOLOGICAL SPACE

Definiton 7: If grey subset family $\mathfrak{I}$ in X satisfies:

(1) $X, 0\in\mathfrak{I}$,

(2) If $A, B\in\mathfrak{I}$, then $A\cap B\in\mathfrak{I}$,

(3) If $A_t (t\in T)\in\mathfrak{I}$, then $\bigcup\limits_{t\in T} A_t\in\mathfrak{I}$.

Then $\mathfrak{I}$ is called a grey topology, and $(X,\mathfrak{I})$ is called a grey topological space. The elements A in $\mathfrak{I}$ are called grey open sets and A^C are called grey closed sets,

where $\bar{\mu}_{A^c}=1-\underline{\mu}_A$, $\quad \underline{\mu}_{A^c}=1-\bar{\mu}_A$.

Definition 8, Let $(x,\mathfrak{I})$ be a grey topological space. $a_{\tilde{\lambda}\Delta}$ is a grey point and A is a grey subset in x. If there exists a grey subset $B\in\mathfrak{I}$, such that $a_{\tilde{\lambda}\Delta}\in B\subseteq A$, then A is called neighbourhood of $a_{\tilde{\lambda}\Delta}$. If there exists a grey subset $B\in\mathfrak{I}$ such that $a_{\tilde{\lambda}\Delta}\triangle B\subseteq A$, then A is called a coincidence field of $a_{\tilde{\lambda}\Delta}$.

We use $\mathcal{U}_{a_{\tilde{\lambda}\Delta}}$ to express all neighbourhood train(or coincidence field train) composed of all neighbourhood(or coincidence field) of $a_{\tilde{\lambda}\Delta}$.

If $A\in\mathcal{U}_{a_{\tilde{\lambda}\Delta}}$ is a open set, then A is called open neighbourhood(or open coincidence field).

Theorem 2. Let $(x,\mathfrak{I})$ be a grey topological space, then neighbourhood train(or coincide field train) of the grey point a in X has the following properties,

(1) $\mathcal{U}_a\neq\emptyset$.

(2) If $A\in\mathcal{U}_a$, then $a\in A$.

(3) If $A,B\in\mathcal{U}_a$, then $A\cap B\in\mathcal{U}_a$.

(4) If $B\in\mathcal{U}_a$, and the grey subset $A\supseteq B$, then $A\in\mathcal{U}_a$.

(5) If $A\in\mathcal{U}_a$, then there exists $B\in\mathcal{U}_a$ such that $A\in\mathcal{U}_b$ for all $b\in B$.

proof, We only prove neighbourhood.

(1) It's obvious, $x\in\mathcal{U}_a$, hence $\mathcal{U}_a\neq\emptyset$.

(2) If $A\in\mathcal{U}_a$, then there exists $B\in\mathfrak{I}$ such that $a\in B\subseteq A$. Hence $a\in A$.

(3) If $A,B\in\mathcal{U}_a$, then there exists $A_1,B_1\in\mathfrak{I}$ such that $a\in A_1\subseteq A$ and $a\in B_1\subseteq B$, hence there exists $A_1\cap B_1\in\mathfrak{I}$ such that $a\in A_1\cap B_1\subseteq A\cap B$, so $A\cap B\in\mathcal{U}_a$.

(4) If $B\in\mathcal{U}_a$, then there exists $B_1\in\mathfrak{I}$ such that $a\in B_1\subseteq B$, hence $a\in B_1\subseteq B\subseteq A$, so $A\in\mathcal{U}_a$.

(5) If $A\in\mathcal{U}_a$, then there exists $A_1\in\mathfrak{I}$ such that $a\in A_1\subseteq A$. Let $B=A_1$, since $A_1\in\mathfrak{I}$ and $a\in A_1$, hence $B\in\mathcal{U}_a$. Since any $b\in B$, $B\in\mathfrak{I}$, then $B\in\mathcal{U}_b$.

From $B \subseteq A$ and (4) we have $A \in \mathcal{U}_b$.

Definition 9. Let $(x, \mathfrak{I})$ be a grey topological space.

(1) $\bigcup\limits_{a \in A} \{B \mid B \in \mathfrak{I}, a \in B \subseteq A\}$ is called the interior of A, usually written $\mathring{A}$.

(2) $\bigcap\limits_{a \in A} \{C \mid C$ is a grey closed set, $a \in A \subseteq C\}$ is calld the closure of A, usualy written $\bar{A}$.

(3) $\bar{A} \cap (\bar{A}^C)$ is called the grey frontier of A, usually written $b(A)$.

Theorem 3 (1) $\mathring{A}$ is the largest grey open set contained in A.

(2) $\bar{A}$ is the smallest grey closed set containing A.

(3) $\bar{A} \supseteq A \cup b(A)$.

Proof. We prove (3) only.

If we want to prove $\bar{A} \supseteq A \cup b(A)$ only need to prove

$\underline{\mu}_{\bar{A}}(x) \geqslant \bar{\mu}_{A \cup b(A)}(x) \quad \forall x \in X.$

$$\bar{\mu}_{A \cup b(A)}(x) = \max\{\bar{\mu}_A(x), \bar{\mu}_{b(A)}(x)\} = \max\{\bar{\mu}_A(x), \bar{\mu}_{\bar{A} \cap (\bar{A}^C)}(x)\}$$
$$= \max\{\bar{\mu}_A(x), \min[\bar{\mu}_{\bar{A}}(x), \bar{\mu}_{\bar{A}^C}(x)]\}$$
$$= \max\{\bar{\mu}_A(x), \min[\bar{\mu}_{\bar{A}}(x), 1 - \underline{\mu}_{\bar{A}}(x)]\}.$$

Since $\bar{A} \supseteq A$, so $\underline{\mu}_{\bar{A}}(x) \geqslant \bar{\mu}_A(x)$, $\forall x \in X$.

Since $\bar{A}$ and $\bar{A}$ coincide to X, $\forall x \in . X$

Hence $\underline{\mu}_{\bar{A}}(x) + \underline{\mu}_{\bar{A}}(x) > 1 (\forall x \in X) \rightarrow \underline{\mu}_{\bar{A}}(x) > 1 - \underline{\mu}_{\bar{A}}(x) \ (\forall x \in X)$.

So $\underline{\mu}_{\bar{A}}(x) \geqslant \bar{\mu}_{A \cup b(A)}(x)$, $\forall x \in X$.

But usually $A \cup b(A) \supseteq A$ can't hold.

Theorem 4 Let $(x, \mathfrak{I})$ be a grey topological space. The grey subset A is the open set if and only if $A = \mathring{A}$.

Theorem 5 Let $(x, \mathfrak{I})$ be a grey topological space. If the grey point $a_{\bar{\lambda}\underline{\lambda}} \in A$, then every coincide field of $a_{\bar{\lambda}\underline{\lambda}}$ coincide to A all.

Proof. $a_{\bar{\lambda}\underline{\lambda}} \in A \rightarrow$ any grey closed sets $B \supseteq A$, there is always $a_{\bar{\lambda}\underline{\lambda}} \in B$. i.e. $\underline{\mu}_B(a) \geqslant \bar{\lambda} \geqslant \underline{\lambda} > 0$,

$\rightarrow$ any grey open sets $C \subseteq A^c$, always holding, $\bar{\mu}_C(a) \leqslant 1 - \underline{\lambda}$,

$\rightarrow$ any grey open sets C satisfies $\bar{\mu}_C(a) > 1 - \underline{\lambda}$ didn't contain in A^c.

$\rightarrow$ any grey open sets C satisfies $\underline{\mu}_C(a)>1-\Delta$ didn't contain in A^C .so C and $(A^C)^C=A$.

$\rightarrow$ any grey open coincide field C of $O_{\bar{\mu}\Delta}$ always coincide with A.

$\rightarrow$ every coincide field of $O_{\bar{\mu}\Delta}$ coincide with A.

III. RELATIONSHIP BETWEEN GREY TOPOLOGICAL SPACE AND FUZZY AND GENERAL TOPOLOGICAL SPACE

1. Let$(x,\mathfrak{J})$ be a grey topological space. If the upper and lower subordinate functions of the grey set A in X are equal, or $\bar{\mu}_A\equiv\underline{\mu}_A$, then change the grey set A in X into the fuzzy set. Hence change $\mathfrak{J}$ into the fuzzy topology and change$(X,\mathfrak{J})$ into the fuzzy topological space.

2. Let$(X,\mathfrak{J})$ be a fuzzy topological space. If subordinate functions of the fuzzy set A $\mu_A\in\{0,1\}$, then change A into the cantor set in X. Hence change $\mathfrak{J}$ into the general topology and change$(X,\mathfrak{J})$ into the general topological space.

Hence general topological space is a particular example of fuzzy topological space. Fuzzy topological space is a particular example of grey topological space.

So grey topological space $\supseteq$ fuzzy topologcal space $\supseteq$ general topological space.

IV COMPACT GREY TOPOLOGICAL SPACE

Definition 10 The grey sets family $\{At\mid t\in T\}$ is called the cover of the grey set B if and if $B\subseteq\bigcup_{t\in T}At$. If At are all grey open set, this is called open cover. If some sub-family is still covered, then this is called subcover.

Definition 11 The grey topological space$(X,\mathfrak{J})$ is called compact if and only if every open cover has finite subcover.

Definition 12 The mapping from the grey topological space $(X_1,\mathfrak{J}_1)$ to the grey topological space$(X_2,\mathfrak{J}_2)$ is called grey continuation if and only if $\forall B\in\mathfrak{J}_2\rightarrow f^{-1}[B]\in\mathfrak{J}_1$.

Theorem 6 The composition mapping of two mappings is also a grey continuous mapping.

Proof, Let grey continuous mapping $f: (X_1, \mathfrak{I}_1) \to (X_2, \mathfrak{I}_2)$,
$g: (x_2, \mathfrak{I}_2) \to (x_3, \mathfrak{I}_3)$ (the grey topological space).
If $\forall C \in \mathfrak{I}_3$, then $(g \circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$.
Since g is a grey continuous mapping, hence $g^{-1}[C] \in \mathfrak{I}_2$.
Since f is a grey continuous mapping, hence $f^{-1}[g^{-1}[C]] \in \mathfrak{I}_1$.
So $(g \circ f)^{-1}[C] \in \mathfrak{I}_1$.
Hence $g \circ f$ is a grey continuous mapping.

Theorem 7 Let f be a grey continuous surjetion from the grey topological space $(X_1, \mathfrak{I}_1)$ to the grey topological space $(X_2, \mathfrak{I}_2)$. If $(X_1, \mathfrak{I}_1)$ is compact, then $(X_2, \mathfrak{I}_2)$ is compact too.

Proof, Let $\{Bt \mid t \in T\}$ be a open cover of X_2.
Since for any $a \in X$ there always holds,
$\underline{\mu}_{\bigcup_{t\in T} f^{-1}[Bt]}(a) = \sup_{t\in T} \underline{\mu}_{f^{-1}[Bt]}(a) = \sup_{t\in T} \underline{\mu}_{Bt}(f(a)) = \underline{\mu}_{\bigcup_{t\in T} Bt}(f(a))$.
Since $\bigcup_{t\in T} Bt \supseteq X_2 \to \underline{\mu}_{\bigcup_{t\in T} Bt}(f(a)) \geqslant \bar{\mu}_{X_2}(f(a)) = 1$.
Hence $\underline{\mu}_{\bigcup_{t\in T} Bt}(f(a)) = 1$. Thus $\underline{\mu}_{\bigcup_{t\in T} f^{-1}[Bt]}(a) = 1$.
Hence $\underline{\mu}_{\bigcup_{t\in T} f^{-1}[Bt]}(a) \geqslant \bar{\mu}_{X_1}(a) \to \bigcup_{t\in T} f^{-1}[Bt] \supseteq X_1$.
So $\{f^{-1}[Bt] \mid t \in T\}$ is a cover of X_1.
Also $f^{-1}[Bt]$ $(t \in T)$ is a grey open set, (actually Bt is a grey open set, f is a grey continuous mapping. So $f^{-1}[Bt]$ $(t \in T)$ is a grey open set too).

Hence $\{f^{-1}[Bt] \mid t \in T\}$ is a open cover of X_1. Since $(X_1, \mathfrak{I}_1)$ is compact, so it has finite subcover $\{f^{-1}[Bt_k] \mid k = 1, 2, \ldots, n\}$.
Also, f is a surjetion, then for any grey subset B of X_2, there always holds $f[f^{-1}[B]] = B$. Hence $\{Bt_k \mid k = 1, 2, \ldots, n\}$ is a cover of X_2. So $(X_2, \mathfrak{I}_2)$ is compact.

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