TOWARDS FUZZY MODELLING IN ECONOMICS (°)

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SUMMARY:

The general methodological framework of fuzzy modelling is considered. Particular emphasis is set on the analytical approach to building fuzzy models, expecially linear fuzzy models. Such models are obtained as a fuzzy extension of usual linear models. Apart from the known extension principle of Zadeh and the so called fuzzy parameter extension, a new model of fuzzy combination is also considered.

KEYWORDS:

fuzzy model, fuzzy relation, extension principle, linguistic modelling, fuzzy mapping.

1. INTRODUCTION

We continuously observe many regularities in the world we live in.

For many reasons we want to express this regularities analytically, by means of mathematical formulae.

These formulae are often called models, or mathematical models.

The usual practice is to assume that quantities under investigation are functionally related.

There are however many situations where these quantities are related in a more obscure manner. Very often it is difficult, or even impossible, to distinguish dependent from independent variates.

Especially in economics, where our understanding of the real phenomena is very poor and incomplete, it seems to be more realistic and more useful, instead of making unrealistic mathematical assumptions about functional dependency, to take the data as they are and to try to represent the relationships among them in such a way that as much information as possible would be preserved.

2. OBSERVATIONS

In order to model any phenomenon by means of mathematical operators it is necessary first of all to distinguish some set of objects underlying the phenomenon under investigation. An object is meant here as any entity showing characteristics that can be measured.

Characteristics are also called variables, quantities or varieties. They are denoted by symbols Y, X^1,X^2.... . For simplicity of exposition only two variables Y and X^1 are considered in this paper, and therefore the latter will be simply denoted as X. If O denotes the set of objects and R stands for real numbers, then the mapping

$$Y : O \to R \qquad (1)$$

is treated as a measurement or observation of Y, which should usually satisfy some conditions (see e.g. [11]).

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The values $Y(o_i)$ for $o_i \in O$ are denoted as y_i and are called the observed values of variable Y. For simplicity, the range of mapping (1) is denoted by the same symbol Y.

Very often it is difficult to perform measurement according to (1), or there isn't any known procedure for that. In such cases the linguistically expressed intensity of the characteristic under investigation can be assigned to each object $o \in O$. The linguistic expressions can be modeled as fuzzy numbers, so that instead of (1) we can have

$$Y : O \rightarrow \mathrm{Fuz}(R) \qquad (2)$$

where Fuz(R) stands for fuzzy numbers.

To the fuzzy measurement of the type (2) we can arrive also performing the usual measurement and then subsequently blurring the obtained results. The motivation for such a procedure is explained below.

The problem of constructing the membership function is a crucial problem for the whole theory of fuzzy sets.

Therefore it seems to be unadequate to start considerations with a proposition like: "suppose the observations are given as fuzzy numbers", which is probably the most often used expression in literature on fuzzy sets.

In this paper, in order to avoid such expression, for obtaining fuzzy numbers from crisp measurements the idea of blurring is applied (see also e.g. [8]).

After performing the measurement we obtain a certain number, say m, but we are sure that this number is one of many other possible numbers.

It seems reasonable to assume that the number so obtained is the most possible or the most credible one. To all other possible numbers grades of possibility or credibility are assigned, such that the more distant the numbers are from the most possible number, the smaller will result the grade. In order to precisely define these grades we introduce here the blurring operator which is treated as a fuzzy subset on the real line depending on three parameters which can be easily determined, e.g. by putting some simple questions to the subject involved in the measurement process.

Suppose the fuzzy subset

$$\phi_r : R \rightarrow [0,1] \qquad (3)$$

depending on fixed parameter $o \leq r < \infty$, satisfies the following conditions:

1) $\phi_r(o) = 1$, for any r
2) $\phi_o(x) = c$, c is constant, for any $x \neq o$
3) if $r_1 \leq r_2$, then ϕ_{r_2} is at least sharp as ϕ_{r_1}, i.e. $r_1 \leq r_2 \Rightarrow \phi_{r_1}(x) \leq' \phi_{r_2}(x)$ for all $x \in R$, where $\leq'$ is the well known sharpeness relation defined as follows:
   $b \leq' a \Leftrightarrow a \leq b \leq 1/2$ or $a \geq b \geq 1/2$.

Such a mapping is called here graduator. The continuous graduator will be called bell-shaped graduator or simply fuzzifier if it satisfies also the following additional conditions:

4) $\phi_r(x) = \phi_r(-x)$,
5) $x_1 < x_2 < 0 \Rightarrow \phi_r(x_1) \leq \phi_r(x_2)$.

The simplest function satisfying the above conditions is the following:

$$\Delta_o(x) = \begin{cases} 1, \text{ if } x = O \\ 0, \text{ otherwise} \end{cases} \tag{4}$$

Other examples of fuzzifiers can be defined as follows;

$$\phi_r^1 (x) = \exp ((\ln 1/2)\, d_r(x)), \tag{5}$$

$$\phi_r^2 (x) = 1/(1 + d_r(x)) , \tag{6}$$

where $d_r(x) = |x|^r$.

For these two fuzzifiers the following holds: $\phi_r^i(1) = 1/2$, $i = 1,2$, for any $0 \leq r < \infty$.
The fuzzifiers with such a condition are referred to as standard fuzzifiers.

It is also easy to check that for any $x \in R$ $\phi_r^i(x) = 1/2$, $i = 1,2$, and in the limit case, when $r \rightarrow \infty$, functions ϕ^1 and ϕ^2 become both the characteristic functions of the interval [-1,1].

The above properties allow the interesting interpretation of the parameter r as a coefficient of fuzzy aversness or as a grade of accuracy attitude. Usually the coefficients are treated as normed to the unit interval, therefore by

$$r = \frac{\alpha}{1-\alpha}, \quad o \leq \alpha \leq 1 \tag{7}$$

the normalized coefficient of fuzzy aversness is defined. In the case of maximal aversness ($\alpha = 1$), fuzzy sets ϕ^i are reduced to the usual interval [-1,1] i.e. membership functions become the characteristics functions. In the case of minimal aversness toward fuzziness, i.e. when $\alpha = 0$, the fuzzy sets ϕ^i become the most fuzzy, i.e. $\phi_o^i(x) = 1/2$ for any $x \in R$.

Having some fuzzifier ϕ let us define the fuzzy set F by the formula:

$$F(x) = \phi\left(\frac{x-m}{s}\right), \; s \geq 0, \tag{8}$$

putting by convention

$$\phi\left(\frac{x-m}{o}\right) = \Delta_o\,(x-m).$$

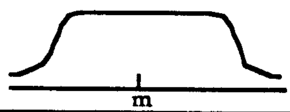
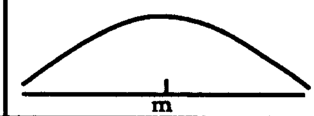
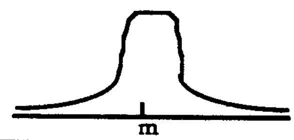
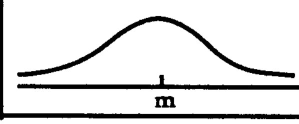
The parameters m and s are called respectively the mode and the spread of the fuzzy set F.

We have adopted here the usual definitions:

- the mode or modal value of F denoted as $\mu(F)$ is the value m such that $F(m) = 1$

- the spread of F denoted as δ(F) is defined to be equal to half of the diameter of the set $\{x \mid F(x) \geq 1/2\}$.

The intuitive meaning of parameters m, r and δ is explained as follows:

spread	fuzziness aversness	
	big	small
big		
small		

We are ready now to present the blurring procedure, mentioned at the beginning of this paragraph.

Suppose a number m is given, obtained by some imprecise measurement process (e.g. by expert judgement), it is reasonable to consider it as a modal value of fuzzy number Y. In order to determine the spread of this number it is enough to have the most uncertain value, i.e. a value k such that F(k) = 1/2, the spread is then egual to the number abs (m - k).

Within the class of the same accuracy attitude we introduce the equivalence relation: $A \equiv B \Leftrightarrow \mu(A) = \mu(B)$ & $\delta(A) = \delta(B)$.

The grade of fuzzy aversness or accuracy coefficient, which is assumed to be fixed for a given subject or for a given measurement process, can be determined by means of some experiments which are not discussed here.

3. MODELS, LINEAR MODELS AND THEIR EXTENSIONS

Suppose crisp measurements of two economic variables X and Y are given in the form of the following sequence of pairs $(x_1, y_1), ..., (x_N, y_N)$. These measurements, also called observations, will be treated here as an observational relation $R_0 = \{(x_1, y_1), ..., (x_N, y_N)\} \subseteq X \times Y$ (see [9]). If there is a strict inclusion $R_0 \subset X \times Y$, it is reasonable to say that variables X and Y interact. In order to account for this interaction the usual praxis is to find some mathematical (analytical) expression which mirrors this interaction.

One of the simplest way consists in finding a function $f : X \to Y$ such that $y_i = f(x_i)$ for $i = 1, ..., N$. If $y_i \neq y_j \Rightarrow x_i \neq x_j$, then a function with such properties certainly exists. Normally we face the situation where condition $y_i \neq y_j \Rightarrow x_i \neq x_j$ is not satisfied. In this case we look for some approximate solution. This means that we look for some relation instead of function, or for a function f such that conditions $f(x_i) = y_i$, $i = 1, ..., N$ are violated as least as possible (assuming of course that function f belongs to some prespecified set of possible approximants).

Suppose for example that we look for a function $G : X \times Y \to R$ such that for all observed values (x_i, y_i):

$$G(x_i, y_i) \geq 0. \tag{9}$$

Function G defines the relation $R_m \subset X \times Y$ which is referred to as a model relation, this relation accounts for the relationship between variables X and Y.

Without any other restriction function G might be defined as follows:

$$G(x,y) = x^2 + y^2 - r^2,$$

where r is big enough to satisfy the condition (9). A particular case of (9) is the model:

$$G(x,y) = 0 \tag{10}$$

which determines some implicit relationship between X and Y.

The most often used model is however an explicit function

$$f : X \rightarrow Y \tag{11}$$

belonging to a prespecified class of mappings. Very often the linear (in a_o, a_1) function:

$$f(x; a_o, a_1)) = a_o \varphi_o(x) + a_1\varphi_1(x) \tag{12}$$

is used, where φ_o and φ_1 are some fixed real functions defined on X.
It means that in this case the set of possible model relations is the following:

$$M = \{R_m \mid (x,y) \in R_m \Leftrightarrow y = a_o \varphi_o(x) + a_1 \varphi_1(x), a_o, a_1 \in R\} . \tag{13}$$

In the simplest case, when $\varphi_o(x) = 1$, $\varphi_1(x) = x$, the following linear model is obtained

$$y = a_o + a_1x. \tag{14}$$

Roughly speaking, to determine the model (12) means finding the coefficients a_o and a_1 such that function (12) satisfies some optimality criterion in fitting the observed xdata R_o. In the ideal case the following system should be satisfied:

$$y_i = f(x_i) \ , \ \text{for } i = 1, ..., N. \tag{15}$$

In practice we look for approximate solution, i.e. we look for a function such that the error measured by distance

$$D(y,z) = \left(\sum_{i=1}^{N} \mid y_i - f(x_i) \mid^r\right)^{1/r}, \qquad r > 0 \tag{16}$$

is as small as possible, where $y = (y_1, ..., y_N)$, $z = (f(x_1), ..., f(x_N))$.

We can work in the same way in the case of fuzzy modelling. Suppose that observations are given now as fuzzy numbers:

$$(X_i, Y_i) \ , \ i = 1, ..., N \tag{17}$$

where $X_i : R \rightarrow [0,1]$ and $Y_i : R \rightarrow [0,1]$.

The fuzzy granular observations (using the terminology of [3]) can be represented by some usual (non fuzzy) function or by a fuzzy function. In the first case it means that the set of models is the following:

$$M = \{f \mid f : X \rightarrow Y\}. \tag{18}$$

From this model set, as previously, we want to choose the best element. As a criterion of optimality the following system of equalities could be chosen:

$$Y_i = F(X_i) \ , \ i = 1, ..., N \ , \tag{19}$$

where fuzzy set $F(X_i)$ is defined according to the known extension principle:

$$F(X_i)(y) = \sup_x X_i(x) \cdot \Delta_0 (y - f(x)).$$

The conditions for the existence of a function f which satisfies (19) are given by Dubois and Prade in [3].

The second type of fuzzy modelling consists in representing the fuzzy granular observations by means of some fuzzy relation. In this case the set of possible models, i.e. fuzzy models, is the family of all fuzzy relations:

$$M = \{R \mid R : X \times Y \rightarrow [0,1]\} . \tag{20}$$

From this set we want to choose a relation R_m such that:

$$Y_i = R_m (X_i) \ , \ i = 1, ..., N \ , \tag{21}$$

where by $R_m (X_i)$ we mean the sup-min composition:

$$R_m (X_i)(y) = \sup_x \min (X_i (x), R_m (x,y)) .$$

The condition for solvability of system (21) is given by Gottwald in [5]. Namely, system (21) has a solution if and only if the fuzzy relation

$$R_m = \bigcap_{i=1}^{N} X_i \circledast Y_i \tag{22}$$

is a solution to this system, where

$$(X_i \circledast Y_i)(x,y) = X_i (x) * Y_i (y)$$

and * stands for Sanchez operator:

$$a * b = \begin{cases} 1 , \text{ if } a \leq b \\ 0 , \text{ otherwise} . \end{cases}$$

It is interesting to note that if there are solutions for (21), then relation (22) is the greatest one.

Other solutions to (21) (if any) are considered in [5].

System (21) can however have no solution, in this case the approximate solution can be looked for.

To reach this aim some metric for fuzzy numbers is needed, which can be defined in various ways.

The simplest, very often used, is the following modification of city metrics:

$$D(A,B) = \sup_{x} | A(x) - B(x) | \tag{23}$$

Instead of (23) the Zadeh's separation

$$D(A,B) = 1 - \sup_{x} \min (A(x), B(x)) \tag{24}$$

can also be used.

The quantity

$$S = \bigvee_{i=1}^{N} D (R_m(X_i), Y_i) \tag{25}$$

can be considered as approximation error.

The problem of fuzzy modelling in this case is trasformed in to the optimization problem:

$$\min_{R \in M} \bigvee_{i=1}^{N} D (R (X_i), Y_i), \tag{26}$$

where M is defined by (20).

In contrast to the problem:

$$\min_{f \in M} \sum_{i=1}^{N} (y_i - f(x_i))^2,$$

where M is defined by (13), there is no efficient method for solving (26).

It is worthy to note that both model spaces (18) and (20) are very general, and no restriction was put neither to the kind of functions nor to the type of relations.

There are however two other possible kinds of fuzzy modelling based on the fuzzy extensions of the set (13).

The first one, roughly speaking, consists in replacing numbers α_0 and α_1 by fuzzy numbers A_0 and A_1, and it is known as fuzzy parameter extension.

The second kind consists in replacing function φ_0 and φ_1 by fuzzy relations Ψ_0 and Ψ_1, and it is called here as fuzzy combination extension.

4. FUZZY PARAMETER EXTENSION MODELS

Suppose that for fixed sets X, Y and K following function is given

$$f : X \times K \rightarrow Y \tag{27}$$

which should be extended to the fuzzy mapping:

$$F : X \times \mathrm{Fuz}(K) \rightarrow \mathrm{Fuz}(Y). \tag{28}$$

Suppose that a fuzzy subset $A \in \mathrm{Fuz}(K)$ is given, treated as fuzzy parameter, then the fuzzy subset $Y_x \in \mathrm{Fuz}(Y)$, which corresponds to fuzzy parameter A and to any argument $x \in X$, is defined as follows (see [1,2]):

$$Y_x(y) = \sup_a \left[A(a) \cdot \Delta_0 \left(y - f(x,a)\right)\right]. \tag{29}$$

Taking for example

$$f(x,a) = a_0 + a_1 x, \tag{30}$$

and assuming $a = (a_0 , a_1)$, $A = (A_0 , A_1)$ and $A\,(a_0 , a_1) = \min\,(A_0(a_0), A_1(a_1))$, then from (29) the following model follows (see [12]):

$$Y_x = A_0 + A_1 \cdot x \tag{31}$$

where A_0 and A_1 are fuzzy numbers, and operations + and $\cdot$ are defined as usually:

$$(A_0 + A_1)\,(z) = \sup_a \min\,(A_0\,(a), A_1\,(z-a)),$$
$$(A \cdot x)\,(z) = A(z/x), \qquad \text{for } x \neq 0.$$

The problem is now to determine the fuzzy coefficients A_0 and A_1, in such a way that model (31) fits in the observed data as well as possible.

In econometrics two methods for estimation are in use: least squares method and maximum likelihood method.

Instead of the least squares - like techniques considered in the previous paragraph, optimization - like techniques will now be discussed. It is known that maximum likelihood estimation requires the assumption about the probability distribution. Analogously, in this and in the next paragraph assumptions about possibility distribution of fuzzy variables will be required.

Suppose, for example, that fuzzy numbers A_0 and A_1 are of the following type:

$$A_i\,(a) = \phi\left(\frac{a - a_i}{s_i}\right), \; i = 0,1 \tag{32}$$

where ϕ is some fuzzifier (discussed in paragraph 2) and a_i, s_i are parameters.

Putting $\phi(z) = e^{-|z|^r}$, $r \geq 0$, fuzzy mapping (31) takes the form:

$$Y_x(y) = e^{-\frac{|y - a_0 - a_1 x|^r}{s_o^r + |s_1 x|^r}} \qquad (33)$$

which is considered as a fuzzy model. The problem is now to determine the parameters occurring in it.

The value $Y_x(y)$ can be interpreted as a grade of connectedness between x and y, or by other words, as a truth value of the proposition: "pair (x,y) is a good model point". Taking into account such an interpretation it seems reasonable to determine the parameters a_0, a_1, s_0, and s_1 in such a way that:

for all i = 1, ..., N, $Y_{x_i}(y_i)$ is as much close to 1, i.e. to maximal admitted value, as possible.

If the quantifier "all" will be transformed in product operator, then the above sentence could be expressed as an optimization problem:

$$\text{maximize} \prod_{i=1}^{N} Y_{x_i}(y_i)$$

with respect to parameters a_0, a_1, s_0, s_1.

If $Y_{x_i}(y_i)$ is defined by (33), then the above maximization problem is equivalent to the following problem:

$$\min_{a,s} \sum_{i=1}^{N} \frac{|y_i - a_0 - a_1 x_i|^r}{s_o^r + |s_1 x_i|^r}, \qquad a = (a_0, a_1)\ ,\ s = (s_0, s_1).$$

If the parameters s_0 and s_1 are fixed, then parameters a_0 and a_1 can be determined by one of the many existing procedures for L_r approximation (see e.g. [4]).

If the parameter s_0 and s_1 are also to be determined, the problem becomes much more complicated.

There are now two possibilities: either minimize numerator with some restriction put on denominator or vice-versa, minimize denominator (standing for fuzziness) with restriction on numerator (standing for fitting).

The latter approach was applied by Tanaka, Uejima and Asai in their interesting paper [12]. This method is briefly presented here and then some modifications are considered.

Suppose that observations are given in the form (x_i, Y_i), i = 1, ..., N, where x_i is a real number and Y_i is a fuzzy number.

Fuzzy numbers Y_i (i = 1, ... ,N) are defined as triangular numbers with mode m = y_i and spread s = l_i (y_i, l_i are given real numbers, $l_i > 0$). In the same way are defined fuzzy coefficients A_0 and A_1, which are to be determined (see [12]).

Model (31) takes the following form [12]:

$$Y_x(y) = 1 - \frac{|y - a_0 - a_1 x|}{s_0 + s_1 |x|}.$$

Instead of maximization of $\prod_{i=1}^{N} Y_{x_i}(y_i)$ with respect to a_o, a_1, s_o and s_1, which is unlimited, paper [12] contains the proposal to minimize the function:

$$f(a_o, a_1, s_o, s_1) = s_o + s_1$$

subject to the following restrictions: $|y_i - a_o - a_1x_i| \leq t\,(s_o + s_1|x_i| - l_i)$, $i = 1, ..., N$; $s_o \geq 0$, $s_1 \geq 0$, where t is a fixed number ($o \leq t \leq 1$), interpreted as a threshold for admissible degree of fitting of the fuzzy model.

As alternative to the method proposed in [12] the following minimization problem might be introduced:

$$\min_{a,s} \sum_{i=1}^{N} |y_i - a_o - a_1x_i|^r$$

with restrictions: $s_o + s_1|x_i| - l_i \leq |y_i - a_o - a_1x_i|$, $i = 1, ..., N$; $s_o \geq 0$, $s_1 \geq 0$.

For $r = 1$ this problem can be solved by any linear programming method, and for $r = 2$ this is a typical quadratic programming problem.

5. FUZZY COMBINATION EXTENSION MODELS

Suppose that function (27) will be expressed as a relation $\rho : (XxY) \times K \rightarrow \{0,1\}$ defined as follows : $\rho(x, y, a) = \Delta_o(y - f(x,a))$.

The extensions of (27) to the mapping $R : Fuz(X \times Y) \times K \rightarrow [0,1]$ is defined as follows:

$$R(x,y,a) = \sup_{x,y} \left[\Psi(x,y) \cdot \Delta_o(y-f(x,a))\right], \tag{35}$$

with $\Psi(x,y)$ fuzzy relation in X x Y.

Let the function (27) be of the form:

$$f(x,a) = a_o\,\varphi_o(x) + a_1\,\varphi_1(x) \tag{36}$$

where φ_o, φ_1 are fixed functions.

Let $\Psi(x,y)$ be defined as the minimum of two fixed fuzzy relations:

$$\Psi_i : X \times Y \rightarrow [0,1], \; i = 0,1,$$

then the extension (35) takes the form:

$$R(x,y) = a_o\,\Psi_o(x,y) + a_1\,\Psi_1(x,y) \tag{37}$$

where $a_i\,\Psi_i(x,y) = \Psi_i(x, y/a_i)$ and operation + is meant as a sup-min composition of two relations

Suppose that

$$\Psi_i(x,y) = \phi\left(\frac{y - \varphi_i(x)}{s_i}\right) \tag{38}$$

where ϕ is some fuzzifier, $\varphi_i(x)$ are given functions and $s_i \geq 0$, then fuzzy combination extension of (36) will take the form:

$$R_m(x,y) = \phi\left(\frac{y - a_o\varphi_o(x)}{a_o\, s_o}\right) + \phi\left(\frac{y - a_1\varphi_1(x)}{a_1\, s_1}\right).$$

Suppose that $\varphi_o(x) = 1$, $\varphi_1(x) = x$ and $\phi(z) = \exp(-|z|^r)$, then the above fuzzy model becomes the following:

$$R_m(x,y) = e^{-\frac{|y - a_o - a_1 x|^r}{|a_o s_o|^r + |a_1 s_1|^r}}. \tag{39}$$

To determine the coefficients, following a similar reasoning as in previous paragraph, we have to solve the problem:

$$\max_a \prod_{i=1}^{N} R_m(x_i, y_i),$$

which is equivalent to

$$\min_a \sum_{i=1}^{N} \frac{|y_i - a_o - a_1 x_i|^r}{|a_o s_o|^r + |a_1 s_1|^r}. \tag{40}$$

Let us consider the particular case for $r = 2$.
We introduce the following notations:

$$A = \begin{bmatrix} -2y \\ -2xy \end{bmatrix},\ a = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix},\ B = \begin{bmatrix} 1 & x \\ x & x^2 \end{bmatrix},$$

$$D = \begin{bmatrix} s_0^2 & 0 \\ 0 & s_1^2 \end{bmatrix},\ c = y^2,$$

then (40) becomes:

$$\min_a \sum_{i=1}^{N} \frac{A_i' a + a' B_i a + c_i}{a' D a},$$

where $A_i' = [2y_i \quad 2x_iy_i]$ and B_i and c_i are similarly defined.
For the minimization some non-linear programming technique can be applied.

6. FUZZY FENCED MODELS

The model is usually considered as some mapping, and the common practice is to put emphasis only on the "law" of this mapping, not taking care of its range and domain. The domain of the model is particularly important (the discussion of this problem could be find in [6]).

Using fuzzy sets methodology it is quite easy to take account of law and domain, both law and domain in the case of fuzzy models being expressed in the same manner i.e. as fuzzy sets. There is remarkable analogy with fuzzy optimization where goals as well as constraints are aggregated into one fuzzy set.

In the framework of fuzzy modelling another important methodological interpretation could be proposed.

It is well known that two approaches to modelling could be distinguished: model oriented and data oriented.

As we will see later on, these two approaches could be considered together.

To this aim, let us consider the observed data as a fuzzy relation R_o.

In the crisp case it was very natural to consider the set of pairs (x_i , y_i) as a relation $R_o \subset X \times Y$.

For the fuzzy case another definition is needed.

Following [3,5] the fuzzy grain is defined here as a fuzzy relation R_o^i:

$$R_o^i (x,y) = X_i(x) \top Y_i(y) ,$$

where $\top$ is some t-norm and X_i, Y_i are fuzzy numbers, which represent the fuzzy observation.

In the case when observations are given by crisp numbers (x_i , y_i), fuzzy grain R_o^i is defined by the formula:

$$R_0^i(x,y) = \phi\left(\frac{x-x_i}{s_i}\right) \top \phi\left(\frac{y-y_i}{s_i}\right) , \ s_i > 0$$

where ϕ is some fuzzifier.

Observed fuzzy relation R_o is defined then as

$$R_o = \mathop{\perp}_{i=1}^{N} R_o^i$$

where $\perp$ stands for some t-conorm.

Suppose that two characteristics X and Y are functionally related by $y = f(x)$.

The model f should be restricted to the observed data and this can be done by constructing the following fuzzy relation R_m restricted (fenced) to the observed data R_o (see [10]):

$$R_r (x,y) = R_o (x,y) * \Delta_o (y - f(x)) ,$$

where $*$ stands for some confluence operator e.g. t-norm. For example, if $f(x) = ax + b$ the above model is nothing else than a fuzzy segment on the plane, treated as fuzzy subset of the following set: $\{(x,y) \mid y = ax + b\}$.

Suppose now we want to restrict fuzzy relation R_o to those points which approximately are lying on the line y = ax + b. Using some fuzzifier, the expression "approximately lying on the line" can be formalized as a fuzzy set $\phi\left(\frac{y - ax - b}{s}\right)$, and the restricted fuzzy model takes the form:

$$R_r(x,y) = R_o(x,y) * \phi\left(\frac{y - ax - b}{s}\right).$$

Geometrically a certain fuzzy segment is formed on the plain, but now not only the length but also the width is fuzzy.

Generally if R_m is any fuzzy model determined by some methods described in previous paragraphs, then it can be restricted to the observed data making confluence of two fuzzy relations:

$$R_r(x,y) = R_o(x,y) * R_m(x,y).$$

If we have no theory for costructing the model R_m, this means there is no reason to distinguish any particular relation R_m, then we put identically $R_m(x,y) \equiv 1$.

In this case we follow an entirely data oriented approach, we trust only in observed data, R_o is our model.

On the contrary if the observed data are so vague that we cannot distinguish any value which could be considered as a more possible than the other, then by putting $R_o(x,y) \equiv 1$ we follow the model oriented approach.

7. Concluding Remarks

J. Johnston, one of the founders of econometric methods, said that the bulk of conventional economic theory postulates exact functional relationships between variables. The most elementary acquaintance with economic data, however, indicates that points do not lie exactly on straight lines or other smooth functions (see [7]).

In order to circumvent these inadequacies, statistical relationships or mean-value relationships were introduced instead of functional relationships between two variables

The other way to describe the observed relationships is to use fuzzy sets methodology.

Econometric theory has firm mathematical foundation but weak empirical confirmation.

Fuzzy sets approach to economic modelling on the contrary, is intuitively very appealing but until now has not been founded mathematicaly.

There are many possibilities of formalization but very little empirically based criteria to choose among them.

In this paper we have discussed some of the possibilities which seem to be more workable than others.

This paper in its spirit is closely related to that of Dubois and Prade [3]. Dubois and Prade distinguished two stand-points of modelling, the analytical and the granular, and the latter is considered in their paper. In our paper we follow the analytical point of view first presented by Chang [2] and then developed in [12].

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