FUZZY SET DESCRIPTION OF PHYSICAL SYSTEMS AND THEIR DYNAMICS

J A R O S Ł A W   P Y K A C Z

Uniwersytet Gdański, Instytut Matematyki, ul. Wita Stwosza 57,
80-952 Gdańsk, Poland.

Abstract: Quantum logic notions are expressed in terms of the fuzzy set theory. The notion of a fuzzy quantum dynamical system is introduced and a version of the Poincaré recurrence theorem is proved.

1. Introduction

Dynamical problems do not belong to the mainstream of the quantum logic approach to the foundations of quantum mechanics. There exist attempts at decsribing temporal changes of states and properties of physical systems within this approach (see, for example, Chapter 23 of Beltrametti and Cassinelli book [1] and references cited therein) but, as it was mentioned in [1], the quantum logic approach is generally a static one. Recently, D. Markechova [2] made an interesting attempt at generalizing the notion of a dynamical system in the sense of the classical probability theory [3] to the so called F-quantum spaces studied by Riečan [4]. However, F-quantum spaces, introduced within the framework of the fuzzy set theory by Piasecki [5] under the name of soft fuzzy σ-algebras, resamble quantum logics only by their general form. Particularly, Zadeh's [6] max and min fuzzy connectives used by Piasecki and Riečan cannot describe quantum logic meet and join and, moreover, the negation in F-quantum spaces is not an orthocomplementation [7,8]. There exists, however possibility of describing quantum logics in the language of the fuzzy set theory. This possibility was mentioned in [1] and developed in [9] by Guz and in [10,7,8,11] by the present author. Therefore, it is worthwhile to apply the idea of Markechova to quantum logics described in fuzzy set theory terms. This is the subject of the present paper.

2. Fuzzy set approach to quantum logics

Def.1. By a *quantum logic* (or simply a *logic*) throughout this paper we mean partially ordered, orthocomplemented, σ-orthocomplete orthomodular set, i.e. a partially ordered set L in which

(i) the least element O and the greatest element I exist,
(ii) the orthocomplementation map $':L\to L$, such that $a''=a$, $a\vee a'=I$, and $a\leq b \Rightarrow b'\leq a'$ is admitted,
(iii) the least upper bound $\bigvee_i a_i$ of any sequence of elements $a_1, a_2, a_3, \ldots$ such that $a_i\leq a_j'$ for $i\neq j$ exists, and
(iv) the orthomodular identity $a\leq b \Rightarrow b = a\vee(a'\wedge b)$ holds.

We would like to warn the reader accustomed to the fuzzy set notation that throughout this paper $a\vee b$ and $a\wedge b$ denote, respectively, the least upper bound (join) and the greatest lower bound (meet) of elements $a,b\in L$ with respect to the given partial order $\leq$ and that they do not denote Zadeh's max and min fuzzy connectives.

Def.2. By a *probability measure* on a logic L we mean a map $m : L \longrightarrow [0,1]$ such that $m(I) = 1$ and $m(\vee_i a_i) = \sum_i m(a_i)$ for any sequence of elements $a_1, a_2, a_3, \ldots$ such that $a_i \leq a_j'$ for $i \neq j$. A set S of probability measures on L is called *full* iff $m(a) \leq m(b)$ for all $m \in S$ implies $a \leq b$.

Elements of a logic are usually called *propositions* or *properties* or *yes-no observables* and it is assumed that they represent properties of a physical system. Probability measures on a logic represent states of a physical system and therefore they are usually called *states* on a logic. If a is a proposition and m is a state then the number $m(a) \in [0,1]$ is interpreted as the probability of obtaining a positive result in an experiment testing a property of a physical system represented by a when this system is in the state represented by m.

Since for any proposition a and for any state m the number $m(a)$ belongs to the unit interval, states can be treated as fuzzy subsets of an universum L, and conversely, propositions can be treated as fuzzy subsets of an universum S. This second possibility allows us to pass to the fuzzy set theory with the aid of the following theorem of Mączyński [12,13].

Theorem. 1. (Mączyński [12], proof in [13]).

(i) If L is a logic with a full set of probability measures S, then each $a \in L$ induces a function $\underline{a} : L \longrightarrow [0,1]$ where $\underline{a}(m) = m(a)$ for all $m \in S$. The set of all such functions $\underline{L} = \{\underline{a} : a \in L\}$ satisfies the following condition :

Orthogonality Postulate: If $\underline{a}_1, \underline{a}_2, \ldots$ is a sequence of functions such that $\underline{a}_i + \underline{a}_j \leq 1$ for $i \neq j$, then there exists $\underline{b} \in \underline{L}$ such that $\underline{b} + \underline{a}_1 + \underline{a}_2 + \ldots = 1$.

$\underline{L}$ equipped with the natural partial order : $\underline{a} \leq \underline{b}$ iff $\underline{a}(m) \leq \underline{b}(m)$ for all $m \in S$ and complementation $\underline{a}' = 1 - \underline{a}$ is isomorphic to L.

(ii) Conversely, if $\underline{L} \subseteq [0,1]^X$ is a set of functions in which the Orthogonality Postulate is satisfied then it is a logic with respect to the natural partial order and complementation. Every point $x \in X$ induces a probability measure m_x on $\underline{L}$ where $m_x(\underline{a}) = \underline{a}(x)$ for all $\underline{a} \in L$ and the set $\{m_x : x \in X\}$ is full.

Thanks to this theorem we see that any logic L with a full set of states S is isomorphic to a family $\mathcal{L}$ of fuzzy subsets of S equipped with the standard fuzzy set inclusion and complementation, and such that membership functions of elements of $\mathcal{L}$ satisfy the Orthogonality Postulate.

The Orthogonality Postulate can be expressed with the aid of Giles' *bold union* $A \cup B$

$$\mu_{A \cup B}(x) = \min(\mu_A(x) + \mu_B(x), 1) , \quad (1)$$

bold intersection $A \cap B$

$$\mu_{A \cap B}(x) = \max(0, \mu_A(x) + \mu_B(x) - 1), \quad (2)$$

and the notion of weakly disjoint sets [14]

$$A \text{ and } B \text{ are } \textit{weakly disjoint} \text{ iff } A \cap B = \emptyset \quad (3)$$

in the following way [10,11]:

Fuzzy Orthogonality Postulate: If $A_1, A_2, \ldots$ is a sequence of pairwise weakly disjoint sets, then $\sum_i \mu_{A_i} \le 1$ and there exists B such that $B = (\bigcup_i A_i)'$.

Let us note that if $A_1, A_2, \ldots$ are pairwise weakly disjoint, equalities (1), (2), and (3) imply that

$$\mu_{\bigcup_i A_i} = \sum_i \mu_{A_i} \quad (4)$$

Def 3. By a *fuzzy quantum logic* we mean any family $L(X)$ of fuzzy subsets of an universum X in which the Fuzzy Orthogonality Postulate holds.

Let us note that the notion of a fuzzy quantum logic defined above would coincide with the notion of a statistical σ-algebra of Guz [9] if only first four axioms listed in [9] were adopted. Therefore, the notion of a fuzzy quantum logic is a little more general than the notion of a statistical σ-algebra.

By the part (ii) of the Mączyński Theorem any fuzzy quantum logic is a traditional quantum logic in which the partial order is given by the standard fuzzy set inclusion

$$A \subset B \text{ iff for all } x \in X \quad \mu_A(x) \le \mu_B(x) \quad (5)$$

and orthocomplementattion is given by the standard fuzzy set negation

$$B = A' \quad \text{iff} \quad \text{for all } x \in X \quad \mu_B(x) = 1 - \mu_A(x). \quad (6)$$

By the part (i) of this theorem any traditional quantum logic with a full set of probability measures is isomorphic to a fuzzy quantum logic. The whole universum X and the empty set $\emptyset$ are, respectively, the greatest and the least element of a fuzzy quantum logic $L(X)$ and we can express the definition of a probability measure (state) on a fuzzy quantum logic in fuzzy set terms in the following way

Def. 4. By a *probability measure (state)* on a fuzzy quantum logic $L(X)$ we mean a mapping $m: L(X) \longrightarrow [0,1]$ such that $m(X)=1$ and $m(\bigcup_i A_i) = \sum_i m(A_i)$ for any sequence of weakly disjoint sets.

Let us note, that by the very definition and by the Mączyński Theorem any fuzzy quantum logic $L(X)$ admits a full set of probability measures induced by points $x \in X$ with the aid of the following formula

$$m_x(A) = \mu_A(x) \quad \text{for all } A \in L(X). \quad (7)$$

However, generally, there can exist probability measures on a fuzzy quantum logic which are not induced by points of the universum X. For example if all membership functions of elements of a fuzzy quantum logic $L(X)$ are integrable on the set X and if we define

$$m(A) = c\int_X \mu_A(x)dx \quad \text{where} \quad c = (\int_X 1dx)^{-1} \quad (8)$$

then $m(X)=1$ and from the formula (4) it follows that

$$m(\bigcup_i A_i) = c\int_X(\sum_i \mu_{A_i}(x))dx = \sum_i m(A_i). \tag{9}$$

for any sequence of pairwise weakly disjoint sets, therefore $m: \mathcal{L}(X) \longrightarrow [0,1]$ is a probability measure on $\mathcal{L}(X)$.

Example 1. The most standard example of a fuzzy quantum logic can be obtained via the Mączyński Theorem from the traditional quantum logic of projectors on a Hilbert space $\mathcal{H}$. Let $\mathcal{P}(\mathcal{H})$ be such a logic and let $S(\mathcal{H})$ denote the set of all density matrices on $\mathcal{H}$. The family $\mathcal{L}(S)$ of all fuzzy subsets of $S(\mathcal{H})$, the membership functions of which are defined by

$$\mu_{\mathcal{P}}(\rho) = Tr(\rho P) \quad \text{for all} \quad \rho \in S(\mathcal{H}) \tag{10}$$

where $\mathcal{P} \in \mathcal{L}(S)$ denotes the fuzzy subset of $S(\mathcal{H})$ generated by the projector $P \in \mathcal{P}(\mathcal{H})$, is a fuzzy quantum logic isomorphic to $\mathcal{P}(\mathcal{H})$. Probability measures on $\mathcal{L}(S)$ generated by density matrices are of the form

$$m_\rho: \mathcal{L}(S) \longrightarrow [0,1], \quad m_\rho(\mathcal{P}) = Tr(\rho P). \tag{11}$$

Example 2. Let X be a topological space and let $\mathcal{B}(X)$ be the Boolean algebra of Borel subsets of X. $\mathcal{B}(X)$ is a fuzzy quantum logic in which all elements are crisp and all traditional probability measures on $\mathcal{B}(X)$ are states in the sense of the Definition 4.

3. Fuzzy quantum dynamical systems

The definition of a fuzzy quantum dynamical system adopted here is essentially the same as the definiton given by Markechova in [2] with the only difference that the notions of F-quantum space and F-state are replaced, respectively, by the notions of a fuzzy quantum logic and a state on such logic.

Def.5. By a *fuzzy quantum dynamical system* (FQDS) we mean a quadruple $(X,\mathcal{L}(X),m,U)$ where $\mathcal{L}(X)$ is a fuzzy quantum logic, m is a state on $\mathcal{L}(X)$ and U is a σ-homomorphism (i.e. a mapping $U: \mathcal{L}(X) \longrightarrow \mathcal{L}(X)$ such that $U(\emptyset)=\emptyset$, $U(A')=U(A)'$ and $U(\vee_i A_i)=\vee_i U(A_i)$ fulfilling the following condition

$$m(U(A)) = m(A) \quad \text{for all} \quad A \in \mathcal{L}(X). \tag{12}$$

The notion of a fuzzy quantum dynamical system is non-void since any quadruple $(X,\mathcal{L}(X),m,I)$ where I is an identity mapping is a FQDS. We shall call such FQDS a *trivial* one. The following example shows that there exist fuzzy quantum dtnamical systems which are not trivial.

Example 3. Let $X=[0,1]$ and let $\mathcal{L}(X) = \langle\emptyset,A,A',X\rangle$, where $\mu_A(x)=x$, be the four-element fuzzy quantum logic. Any point $x \in X$ induces a probability measure on $\mathcal{L}(X)$ by the equality (7) and the mapping $U: \mathcal{L}(X) \longrightarrow \mathcal{L}(X)$ defined by $U(\emptyset)=\emptyset$, $U(X)=X$, $U(A)=A'$, $U(A')=A$ is a homomorphism of $\mathcal{L}(X)$ onto $\mathcal{L}(X)$. It is easy to notice that the quadruple $(X,\mathcal{L}(X),m,U)$ is a FQDS if and only if $x=1/2$ and that it is not a trivial one.

Example 4. Let X and $\mathcal{B}(X)$ be as in Example 2 and let $(X,\mathcal{B}(X),\mu,T)$ be a dynamical system in the sense of the classical probability theory [3]. Then $(X,\mathcal{L}(X),\mu,T)$ is a fuzzy quantum dynamical system as well.

Dynamics described with the aid of fuzzy quantum dynamical systems can be viewed as the Heisenberg picture since homomorphism

U acts on propositions, not on states.

Let us finish with a version of Poincaré Recurrence Theorem. This theorem already exists in the literature in various approaches: the classical one (see, for example, [3]), the quantum logic approach [15] and in the F-quantum space approach [16]. Proofs are based on the same idea expressed, respectively, in terms of suitable standard or non-standard probability measures.

Theorem 2. Let $(X, \mathcal{L}(X), m, U)$ be a fuzzy quantum dynamical system such that $\mathcal{L}(X)$ is a lattice. Then for all $A \in \mathcal{L}(X)$ we have

$$m(A \wedge (\vee_i U^i A)') = 0. \tag{13}$$

Proof. Let $B = A \wedge (\vee_i U^i A)'$. We shall prove first that B, UB, $U^2 B, \ldots$ are pairwise weakly disjoint. Actually, if $k \neq l$, then

$$U^k B = U^k(A \wedge (\vee_i U^i A)') = \wedge_i (U^k A \wedge U^{i+k} A') \tag{14}$$

while

$$(U^l B)' = (U^l(A \wedge (\vee_i U^i A)'))' = \vee_i (U^l A' \vee U^{i=l} A). \tag{15}$$

Thus, we see that

$$U^k B \leq (U^l B)' \tag{16}$$

which means that

$$\mu_{U^k B} \leq 1 - \mu_{U^l B} \tag{17}$$

or, equivalently,

$$\mu_{U^k B} + \mu_{U^l B} - 1 \leq 0 \tag{18}$$

and therefore, from (2) and (3), we conclude that $U^k B$ and $U^l B$ are weakly disjoint. Now, since probability measures on fuzzy quantum logics are additive on weakly disjoint sets and the homomorphism U is m-invariant, we can apply the standard argumentation: since

$$m(\vee_i U^i B) = \sum_i m(U^i B) = \sum_i m(B) \leq 1, \tag{19}$$

we conclude that $m(B) = 0$.

Acknowledgments

The author is greatly indebted to Dr Dagmar Markechova for providing him with the manuscript of [2]. Financial support of Polish Ministry of Higher Education (Project Nr. RP I 10) is gratefully acknowledged.

References

[1] Beltrametti, E.G. and Cassinelli, G. The Logic of Quantum Mechanics, Addison-Wesley, Reading 1981.

[2] Markechova, D. Homomorphism and conjugation of fuzzy dynamical systems, submitted to BUSEFAL.

[3] Fomin, S.W., Kornfeld, I.P. and Sinaj, J.G. Ergodic Theory, Nauka, Moscow, 1890 (in Russian).

[4] Riečan, B. A new approach to some notions of statistical quantum mechanics, BUSEFAL, 35 (1988), 4-6.

[5] Piasecki, K. Probability of fuzzy events defined as denumerable additivity measure, Fuzzy Sets and Systems, 17 (1985), 271-284.

[6] Zadeh, L.A. Fuzzy sets, Information and Control, 8 (1965), 338-353.

[7] Pykacz, J. Quantum logics and soft fuzzy probability spaces, BUSEFAL, 32 (1987), 150-157.

[8] Pykacz, J. Quantum physics, fuzzy sets and the problem of connectives, to be published in the Proceedings of the International Meeting on Fuzzy Set Theory, Poznan, September 1988, J. Albrycht and K. Piasecki, eds.

[9] Guz, W. Fuzzy σ algebras of physics, International Journal of Theoretical Physics, 24 (1985), 481-493.

[10] Pykacz, J. Quantum logics as families of fuzzy subsets of the set of physical states, Preprints of the Second IFSA Congress, Tokyo, July 20-25 1987, vol. 2, 437-440.

[11] Pykacz, J. Probability measures in the fuzzy set approach to quantum logics, in : Proceedings of the First Winter School on Measure Theory, Liptovsky Jan, January 10-15, 1988. A. Dvurecenskij and S. Pulmannova, eds. Bratislava, 1988.

[12] Mączyński, M.J. The orthogonality postulate in axiomatic quantum mechanics, Int. J. Th. Phys. 8 (1973), 353-360.

[13] Mączyński, M.J. Functional properties of quantum logics, Int. J. Th. Phys. 11 (1984), 149-156.

[14] Giles, R. Lukasiewicz logic and fuzzy set theory, Int. J. Man-Machine Studies, 8 (1976), 313-327.

[15] Dvurečenskij, A. On some properties of transformations of a logic, Mathematica Slovaca, 26 (1976), 131-137.

[16] Dvurečenskij, A. and Tirpakova, A. Ergodic theory on fuzzy quantum spaces. to be published in the Proceedings of the International Meeting on Fuzzy Set Theory, Poznań, September 1988, J. Albrycht and K. Piasecki, eds.