ISOMORPHISM AND CONJUGATION OF FUZZY DYNAMICAL SYSTEMS

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Using fuzzy sets theory ideas [1] we introduce the notion of fuzzy dynamical system which generalizes the notion of dynamical system of the Kolmogorov classical model of probability theory [2]. The notions of isomorphism and conjugation of fuzzy dynamical systems defined here are generalizations of the above notions of classical probability theory [3]. Connections between all the above notions are presented here, too.

1. BASIC DEFINITIONS AND NOTATIONS

Definition 1.1. By an F-quantum space we mean a couple (X,M), where X is a non-empty set and M is a subset of $\langle 0,1\rangle^X$ satisfying the following conditions:

(1.1) If $1(x) = 1$ for any $x \in X$, then $1 \in M$.

(1.2) If $f \in M$, then $f' = 1 - f \in M$.

(1.3) If $f_n \in M$ $(n = 1,2,\dots)$, then $\bigvee_{n=1}^{\infty} f_n := \sup_n f_n \in M$.

(1.4) If $1/2(x) = 1/2$ for any $x \in X$, then $1/2 \notin M$.

The system M is called in the fuzzy theory a soft fuzzy σ-algebra [4]. This structure has been suggested by Riečan ([5], [6]) as an alternative model for quantum mechanics. The set M may be regarded as a partially ordered set in which the relation $\leq$ is defined in the following way: $f \leq g$ iff $f(x) \leq g(x)$ for each $x \in X$. If we define $\bigwedge_n f_n := \inf_n f_n$, then the meet $\wedge$ and the join $\vee$ are related to each other by simple relations: $1 - \bigvee_n f_n = \bigwedge_n (1 - f_n)$, $1 - \bigwedge_n f_n = \bigvee_n (1 - f_n)$, for any sequence $\{f_n\}_n \subset M$. In accordance with the theory of quantum logics, we say that two elements $f, g \in M$ are orthogonal (we write $f \perp g$), if $f \leq 1 - g$.

Definition 1.2. By an F-state on an F-quantum space (X,M) we mean a mapping $m : M \to \langle 0,1 \rangle$ satisfying the following conditions:

(1.5) $m(f \vee (1 - f)) = 1$ for every $f \in M$.

(1.6) If $\{f_n\}_{n=1}^{\infty}$ is a sequence of pairwise orthogonal fuzzy subsets from M, then $m(\bigvee_{n=1}^{\infty} f_n) = \sum_{n=1}^{\infty} m(f_n)$.

Example 1.1. Let $(X, \mathcal{S}, P)$ be a probability space. Put $M = \{\chi_A ; A \in \mathcal{S}\}$, where χ_A is the characteristic function of the set $A \in \mathcal{S}$. Then (X,M) is an F-quantum space and the mapping $m : M \to \langle 0,1 \rangle$ defined by $m(\chi_A) = P(A)$ is an F-state on (X,M).

2. FUZZY DYNAMICAL SYSTEMS

In the following we shall work with this notion:

Definition 2.1. By a fuzzy dynamical system we shall mean the quadruple $(X,M,m,\mathcal{U})$, where (X,M) is an F-quantum space, m is an F-state on (X,M) and $\mathcal{U} : M \to M$ is a σ-homomorphism fulfilling the following condition:

(2.1) $m(\mathcal{U}f) = m(f)$ for every $f \in M$.

Recall that $\mathcal{U} : M \to M$ is a σ-homomorphism, if $\mathcal{U}(f') = 1 - \mathcal{U}(f)$ and $\mathcal{U}(\bigvee_{n=1}^{\infty} f_n) = \bigvee_{n=1}^{\infty} \mathcal{U}(f_n)$ for every $f \in M$ and any sequence $\{f_n\}_{n=1}^{\infty} \subset M$.

Example 2.1. Let $(X, \mathcal{S}, P, T)$ be a dynamical system in the sense of the classical probability theory. If we define (X,M) and m as is Example 1.1 and the mapping $\mathcal{U} : M \to M$ by the equality $\mathcal{U}(\chi_A) = \chi_{T^{-1}(A)}$, then the quadruple $(X,M,m,\mathcal{U})$ is a fuzzy dynamical system. In this case we shall say, that fuzzy dynamical system $(X,M,m,\mathcal{U})$ is induced by $(X,\mathcal{S},P,T)$.

Example 2.2. Let any F-quantum space (X,M) and any F-state m on (X,M) be given. Let $T : X \to X$ is an m-preserving transformation, i.e. $f \in M$ implies $f \circ T \in M$ and $m(f \circ T) = m(f)$. If we define the mapping $\mathcal{U} : M \to M$ by the equality

(2.2) $\mathcal{U}(f) = f \circ T$ for any $f \in M$,

then the quadruple $(X,M,m,\mathcal{U})$ is a fuzzy dynamical system. The mapping $\mathcal{U}$ defined by (2.2) is so-called Koopman operator. By a fuzzy dynamical system induced by pointed mapping T we shall mean the quadruple $(X,M,m,\mathcal{U})$, where $\mathcal{U}$ is defined by (2.2) and this system we denote by (X,M,m,T). The fuzzy dynamical system $(X,M,m,\mathcal{U})$ from Example 2.1 is induced by pointed mapping T.

3. ISOMORPHISM AND CONJUGATION OF FUZZY DYNAMICAL SYSTEMS

Let two fuzzy dynamical systems $(X_1, M_1, m_1, \mathcal{U}_1)$, $(X_2, M_2, m_2, \mathcal{U}_2)$ be given.

Definition 3.1. We say that two fuzzy dynamical systems $(X_1, M_1, m_1, \mathcal{U}_1)$, $(X_2, M_2, m_2, \mathcal{U}_2)$ are weakly isomorphic, if there exist fuzzy sets $f_1 \in M_1$, $f_2 \in M_2$ such that $m_1(f_1) = m_2(f_2) = 1$ and a bijective mapping $\Psi : \{f \in M_1;\ f \leq f_1\} \to \{f \in M_2;\ f \leq f_2\}$ fulfilling the following conditions:

(3.1) Ψ preserves the operations, i.e. $\Psi(\bigvee_{n=1}^{\infty} f_n) = \bigvee_{n=1}^{\infty} \Psi(f_n)$, $\Psi(f^{\prime}) = (\Psi(f))^{\prime}$ for any $f \in N_1$ and any sequence $\{f_n\}_{n=1}^{\infty} \subset N_1$.

(3.2) For every $f \in M_1$, $f \leq f_1$, it holds $m_1(f) = m_2(\Psi(f))$.

(3.3) The diagram
$$\begin{array}{ccc} N_1 & \xrightarrow{\mathcal{U}_1} & N_1 \\ \Psi\downarrow & & \downarrow\Psi \\ N_2 & \xrightarrow[\mathcal{U}_2]{} & N_2 \end{array}$$

is commutative, i.e. $\varphi(\mathcal{U}_1(f)) = \mathcal{U}_2(\varphi(f))$ for every $f \in N_1$, where $N_1 = \{f \in M_1;\ f \leq f_1\}$, $N_2 = \{f \in M_2;\ f \leq f_2\}$.

In particular, we define the notions of isomorphism and strongly isomorphism of fuzzy dynamical systems (X_1, M_1, m_1, T_1), (X_2, M_2, m_2, T_2) induced by pointed mapping T_1, T_2, respectively.

Definition 3.2. We say that two fuzzy dynamical systems (X_1, M_1, m_1, T_1), (X_2, M_2, m_2, T_2) are isomorphic, if there are fuzzy sets $f_1 \in M_1$, $f_2 \in M_2$ such that $m_1(f_1) = m_2(f_2) = 1$, $f_1 \circ T_1 = f_1$, $f_2 \circ T_2 = f_2$, and a bijective mapping φ: Ker $f_1 \to$ Ker f_2 fulfilling the following conditions :

(3.4) Let $f \leq f_1$. Then $f \in M_1$ iff $f \circ \varphi^{-1} \in M_2$.

(3.5) For every $f \in M_1$ such that $f \leq f_1$ it holds $m_1(f) = m_2(f \circ \varphi^{-1})$.

(3.6) The diagram

$$\begin{array}{ccc} \text{Ker } f_1 & \xrightarrow{T_1} & \text{Ker } f_1 \\ \varphi \downarrow & & \downarrow \varphi \\ \text{Ker } f_2 & \xrightarrow[T_2]{} & \text{Ker } f_2 \end{array}$$

is commutative, i.e. $\varphi(T_1(x)) = T_2(\varphi(x))$ for each $x \in$ Ker f_1.

Here Ker f denotes $\{x \in X;\ f(x) = 1\}$.

It may be proved the following theorem.

Theorem 3.1. Two dynamical systems $(X_1, \mathcal{S}_1, P_1, T_1)$, $(X_2, \mathcal{S}_2, P_2, T_2)$ are isomorphic in the classical sense if and only if fuzzy dynamical systems (X_1, M_1, m_1, T_1), (X_2, M_2, m_2, T_2) induced by $(X_1, \mathcal{S}_1, P_1, T_1)$, $(X_2, \mathcal{S}_2, P_2, T_2)$, respectively, are isomorphic.

Definition 3.3. We say that two fuzzy dynamical systems (X_1, M_1, m_1, T_1), (X_2, M_2, m_2, T_2) are strongly isomorphic, if there exists a bijective mapping $\varphi: X_1 \to X_2$ fulfilling the following conditions:

(3.7) $f \in M_1$ iff $f \circ \varphi^{-1} \in M_2$.

(3.8) $m_1(f) = m_2(f \circ \Psi^{-1})$ for any $f \in M_1$.

(3.9) The diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & X_1 \\ \Psi \downarrow & & \downarrow \\ X_2 & \xrightarrow[T_2]{} & X_2 \end{array}$$

is commutative, i.e. $\Psi(T_1(x)) = T_2(\Psi(x))$ for every $x \in X_1$.

It is evident that if $(X_1,M_1,m_1,T_1),(X_2,M_2,m_2,T_2)$ are strongly isomorphic, then they are isomorphic, too.

Theorem 3.2. Let (X_1,M_1,m_1,T_1), $X_2,M_2, m_2,T_2)$ are strongly isomorphic. Then they are weakly isomorphic, too.

Proof. Let $\Psi: X_1 \to X_2$ represents a strongly isomorphism of systems (X_1,M_1,m_1,T_1), (X_2,M_2,m_2,T_2). Then the mapping $\Psi: M_1 \to M_2$ defined by the equality $\Psi(f) = f \circ \Psi^{-1}$ represents a weakly isomorphism of above systems.

The following considerations shall tend to introduction of the conjugation of fuzzy dynamical systems. To this we shall need the some auxiliary assertions and definitions.

Let be any F-quantum space (X,M) given. Let m be F-state on (X,M). In the set M we define the relation $\sim$ in the following way: for every $f,g \in M$, $f \sim g$ iff $m(f \triangle g) = 0$, where $f \triangle g = (f \wedge g') \vee (f' \wedge g)$ is a symmetrical difference of fuzzy sets f,g. Put $[f] = \{g \in M;\ m(f \triangle g) = 0\}$ for any $f \in M$. If $f_1,f_2 \in [f]$, then $m(f_1) = m(f_2)$. In the system $[M] = \{[f];\ f \in M\}$ one can define the relation $\leq$ in the following way : for any $[f]$, $[g] \in [M]$; $[f] \leq [g]$ iff $m(f \wedge g')=0$. The couple $([M], \leq)$ is a partially ordered set with the minimal element $[0]$ and the maximal element $[1]$.

We obtain immediately the following lemmas.

Lemma 3.1. Let any F-quantum space (X,M) and any F-state m on (X,M) be given. Then $[M]$ is a Boolean σ-algebra.

Proof. $\left[\bigvee_{n=1}^{\infty} f_n\right]$ is a least upper bound of a sequence $\{[f_n]\}_{n=1}^{\infty} \subset [M]$, i.e. holds the equality $\bigvee_{n=1}^{\infty} [f_n] = \left[\bigvee_{n=1}^{\infty} f_n\right]$. Further $[f] \wedge [g] = [f \wedge g]$. For any $[f] \in [M]$ we obtain. $[f] \wedge [f'] = [f \wedge f'] = [0]$ and $[f] \vee [f'] = [f \vee f'] = [1]$ and hence $[f]' = [f']$.

Lemma 3.2. If we define the function $[m]$ by the equality $[m]([f]) = m(f)$, for any $[f] \in [M]$, then $[m]$ is a probability measure on the Boolean σ-algebra $[M]$, i.e. $[m]([0]) = 0$, $[m]([1]) = 1$, $[m] \geq 0$ and $[f_n] \wedge [f_m] = [0]$ $(n \neq m)$ implies $[m]\left(\bigvee_{n=1}^{\infty} [f_n]\right) = \sum_{n=1}^{\infty} [m]([f_n])$.

Lemma 3.3. Let any fuzzy dynamical system $(X, M, m, \mathcal{U})$ be given. Then the mapping $[\mathcal{U}] : [M] \to [M]$ defined by $[\mathcal{U}]([f]) = [\mathcal{U}(f)]$ is a σ-homomorphism on the Boolean σ-algebra $[M]$.

Definition 3.4. Two fuzzy dynamical systems $(X_1, M_1, m_1, \mathcal{U}_1)$, $(X_2, M_2, m_2, \mathcal{U}_2)$ are called conjugate, if there exists a bijective mapping $\phi : [M_1] \to [M_2]$ fulfilling the following conditions:

(3.10) ϕ preserves the lattice operations, i.e. $\phi\left(\bigvee_{n=1}^{\infty} [f_n]\right) = \bigvee_{n=1}^{\infty} \phi([f_n])$, $\phi([f]') = (\phi([f]))'$.

(3.11) $[m_2](\phi([f])) = [m_1]([f])$ for every $[f] \in [M_1]$.

(3.12) The diagram

$$\begin{array}{ccc} [M_1] & \xrightarrow{[\mathcal{U}_1]} & [M_1] \\ \phi \downarrow & & \downarrow \phi \\ [M_2] & \xrightarrow[{[\mathcal{U}_2]}]{} & [M_2] \end{array}$$

is commutative, i.e. for each $[f] \in [M_1]$ it holds $\phi([\mathcal{U}_1]([f])) = [\mathcal{U}_2](\phi([f]))$.

Theorem 3.3. Let $(X_1, M_1, m_1, \mathcal{U}_1)$, $(X_2, M_2, m_2, \mathcal{U}_2)$ are weakly isomorphic fuzzy dynamical systems. Then they are conjugate, too.

Proof. Let $\Psi : \{f \in M_1;\ f \leq f_1\} \to \{f \in M_2;\ f \leq f_2\}$

be a mapping representing a weakly isomorphism of systems $(X_1, M_1, m_1, \mathcal{U}_1)$, $(X_2, M_2, m_2, \mathcal{U}_2)$. Then the mapping $\emptyset : [M_1] \to [M_2]$ defined by the equality $\emptyset([f]) = [\Psi(f)]$, for any $[f] \in [M_1]$, represents the conjugation of above systems.

By means of Theorems 3.2 and 3.3 we obtain the next theorem.

Theorem 3.4. Let (X_1, M_1, m_1, T_1), (X_2, M_2, m_2, T_2) are strongly isomorphic fuzzy dynamical systems. Then they are conjugate, too.

Theorem 3.5. Two dynamical systems $(X_1, \mathcal{Y}_1, P_1, T_1)$, $(X_2, \mathcal{Y}_2, P_2, T_2)$ are conjugate in the sense of the classical probability theory if and only if are conjugate the fuzzy dynamical systems (X_1, M_1, m_1, T_1), (X_2, M_2, m_2, T_2) induced by $(X_1, \mathcal{Y}_1, P_1, T_1)$, $(X_2, \mathcal{Y}_2, P_2, T_2)$, respectively.

Remark 3.1. Reverse assertion of the Theorem 3.4 is not valid, in general. From the conjugation of fuzzy dynamical systems does not implies their strongly isomorphism. Namely, from the classical probability theory is known, that there are the dynamical systems $(X_1, \mathcal{Y}_1, P_1, T_1)$, $(X_2, \mathcal{Y}_2, P_2, T_2)$ such that they are conjugate and they are not isomorphic (in the classical sense). By the preceding theorem the fuzzy dynamical systems (X_1, M_1, m_1, T_1), (X_2, M_2, m_2, T_2) induced by $(X_1, \mathcal{Y}_1, P_1, T_1)$, $(X_2, \mathcal{Y}_2, P_2, T_2)$, respectively, are conjugate. Since $(X_1, \mathcal{Y}_1, P_1, T_1)$, $(X_2, \mathcal{Y}_2, P_2, T_2)$ are not isomorphic, are not strongly isomorphic even the systems (X_1, M_1, m_1, T_1), (X_2, M_2, m_2, T_2).

Analogously as in classical case holds the following theorem:

Theorem 3.6. If (X_1, M_1, m_1, T_1), (X_2, M_2, m_2, T_2) are isomorphic, then they are conjugate, too.

Proof. Let Ψ : Ker $f_1 \to$ Ker f_2 represents an isomorphism of systems (X_1, M_1, m_1, T_1), (X_2, M_2, m_2, T_2), where $f_1 \in M_1$,

$f_2 \in M_2$, $m_1(f_1) = m_2(f_2) = 1$ and $f_1 \circ T_1 = f_1$, $f_2 \circ T_2 = f_2$. Then the mapping $\emptyset : [M_1] \to [M_2]$ defined by $\emptyset([f]) = [(f \wedge f_1) \circ \Psi^{-1}]$, for any $[f] \in [M_1]$, represents the conjugation of above systems.

In the similarly way as in Remark 3.1 we obtain that reverse assertion of the Theorem 3.6 is not valid, in general.

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