On the Variance of a Fuzzy Random Variable **

Li Shoumei

Department of Mathematics

Hebei University

Baoding, Hebei, China

In this paper, we shall introduce the concept of the variance of a fuzzy random variable based on [3, 4], and give formulas to calculus of the expected value and the variance of a fuzzy random variable taking in $\mathcal{F}(R^1)$.

Keywords: Fuzzy random variable, The expectation and variance of a fuzzy random variable

L.P.Madeh [3] introduced the concept of a fuzzy random variable and studied the expectation of it by using random variables set theory [1, 2, 5]. Li [4] further discussed the properties of fuzzy random variables. We shall investigate the variance of a fuzzy random variable based on [3, 4] as follows.

Let $(\Omega, \mathcal{A}, P)$ be a probability space where P is a nonatomic probability measure, R^n n-dimensional Euclidean space, $\mathcal{B}$ the set of all Borel subsets of R^n and $K_0(R^n)$ the set of all nonempty compact subsets of R^n.

<u>Definition 1</u> [4] A fuzzy subset $u: R^1 \longrightarrow [0, 1]$ is called a fuzzy number, if

(1) $\forall \alpha \in (0, 1]$, $u_\alpha \triangleq \{x \in R^1: u(x) \geq \alpha\}$ is a closed interval;

(2) $u_1 \triangleq \{x \in R^1: u(x)=1\} \neq \phi$.

A Fuzzy subset $u: R^n \longrightarrow [0, 1]$ is called an extended fuzzy number,

** This term is supported by NNSFC.

if

(1) $\forall \alpha \in (0, 1]$, $u_\alpha \triangleq \{x \in R^n : u(x) \geq \alpha\}$ is compact;

(2) $u_1 \triangleq \{x \in R^n : u(x) = 1\} \neq \phi$.

$\mathcal{F}_0(R^n)$ denotes the set of all extended fuzzy numbers, and $\mathcal{F}(R^1)$ the set of all fuzzy numbers.

Definition 2 [3] A fuzzy random variable is a function $X: \Omega \longrightarrow \mathcal{F}_0(R^n)$ such that

$$\{(w, x): x \in X_\alpha(w)\} \in \mathcal{A} \times \mathcal{B} \quad , \text{ for every } \alpha \in (0, 1]$$

where $X_\alpha(w) = \{x \in R^n : X(w)(x) \geq \alpha\}$.

The equivalent definition of a fuzzy random variable was given in [4] .

Property 1 [4] X is a fuzzy random variable if and only if for every $\alpha \in [0, 1]$, for every $A \in \mathcal{B}_0(K_0)$,

$$\{ w \in \Omega ; X_\alpha(w) \cap A \neq \phi \} \in \mathcal{A}$$

where $\mathcal{B}(K_0)$ denotes the set of all Borel σ-algebras of subsets of $K_0(R^n)$.

Property 2 [4] Let X_i be a fuzzy random variable, and $a_i \in R^1$, $i = 1, \cdots, m$. Then the both $\sum_{i=1}^{m} a_i X_i$ and $\prod_{i=1}^{m} X_i$ are fuzzy random variables, and for every $\alpha \in [0, 1]$

$$(\sum_{i=1}^{m} a_i X_i)_\alpha = \sum_{i=1}^{m} a_i (X_i)_\alpha , \quad (\prod_{i=1}^{m} X_i)_\alpha = \prod_{i=1}^{m} (X_i)_\alpha .$$

Definition 3 [3] The expected value of the fuzzy random variable of X, denoted by E(X), is the extended number $v \in \mathcal{F}_0(R^n)$ such that

$$v_\alpha = \{ E(f): f \text{ is a P-integral section of } X_\alpha \} \triangleq E(X_\alpha),$$

for every $\alpha \in [0, 1]$.

Now we define the variance of X as follows.

Definition 4 Let X^2 be integrably bounded [3] , $E((X-E(X))^2)$ is called the variance of X, denoted by $D^2(X)$.

The existent and uniquesness of $D^2(X)$ can be obtained by property 2 and theorem 3.1 in [3] .

In general case, the computations of the expected value and variance of X are complex just as L.P.Madeh pointed out. The following we limit a fuzzy random variable to taking in $\mathcal{F}(R^1)$.

<u>Theorem</u>: Let X^2 be integrably bounded, then we have

$$D^2(X) = E(X^2) - (E(X))^2 \quad .$$

<u>Proof</u>: We first prove the property:

$$E(X_1+X_2) = E(X_1) + E(X_2)$$

where X_i is any fuzzy random variable, i = 1, 2.

Let $(X_i)_\alpha(w) = [a_i(w,\alpha), b_i(w,\alpha)]$, for every $\alpha \in (0, 1]$ and for every $w \in \Omega$, i = 1, 2. Then for some w , $a_i(w,\alpha)$ is an increasing function of α and $b_i(w,\alpha)$ is a decreasing function of α .

For some $\alpha \in (0, 1]$, let

$$\min (X_i)_\alpha : \Omega \longrightarrow R^1$$
$$w \longmapsto \min(X_i)_\alpha(w) = a_i(w,\alpha) \quad .$$

For every c R^1,

$$\{ w \in \Omega: \min (X_i)_\alpha < c \} = (\bigcup_{\substack{q \in Q \\ q > c}} \{ w \in \Omega : (X_i)_\alpha(w) \cap \{q\} \neq \phi \})$$
$$\cap \{ w \in \Omega : (X_i)_\alpha(w) \cap \{c\} \neq \phi \}$$

where Q is the set of all ratinal numbers.

We have $\{w:\min(X_i)_\alpha < c\} \in \mathcal{A}$ according to the equivalent definition of a fuzzy random variable, a.e. the function $\min(X_i)_\alpha$ measurable about $\mathcal{B}$. Then $\min(X_i)_\alpha$ is a P-integrable section of

$(X_i)_\alpha$ because of $a_i(w,\alpha) \in [a_i(w,\alpha),\ b_i(w,\alpha)]$.

Similarly, the function

$$\max (X_i)_\alpha : \Omega \longrightarrow R^1$$
$$w \longmapsto \max (X_i)_\alpha(w) = b_i(w,\alpha)$$

is also P-integral section of $(X_i)_\alpha$.

It is obvious that for every P-integral section f of $(X_i)_\alpha$, we have $E(\min(X_i)_\alpha) \le E(f) \le E(\max(X_i)_\alpha)$. So we can obtain:

$$E((X_i)_\alpha) = [E(\min(X_i)_\alpha),\ E(\max(X_i)_\alpha)]$$
$$= [E(a_i(w,\alpha)),\ E(b_i(w,\alpha))]$$

then

$$(E(X_1) + E(X_2))_\alpha = (E(X_1))_\alpha + (E(X_2))_\alpha$$
$$= [E(a_1(w,\alpha))\ E(b_1(w,\alpha))]$$
$$+ [E(a_2(w,\alpha)),\ E(b_2(w,\alpha))]$$
$$= [E(a_1(w,\alpha)) + E(a_2(w,\alpha)),\ E(b_1(w,\alpha)) + E(b_2(w,\alpha))]$$
$$= [E(a_1(w,\alpha) + a_2(w,\alpha)),\ E(b_1(w,\alpha) + b_2(w,\alpha))]$$
$$= (E(X_1+X_2))_\alpha$$

Similarly, we have $E(\alpha X) = \alpha E(X)$, for every $\alpha \in R^1$.

Then

$$D(X^2) = E((X-E(X)^2)$$
$$= E(X^2 - 2E(X)X + (E(X))^2)$$
$$= E(X^2) - 2E(X)\ E(X) + (E(X))^2$$
$$= E(X^2) - (E(X))^2$$

By the proof of the theorem, we know, if $X_\alpha(w) = [a(w,\alpha),\ b(w,\alpha)]$ then

$$E(X) = \bigcup_{\alpha\in(0,1]} \alpha\cdot[E(a(w,\alpha)),\ E(b(w,\alpha))]$$

and

$$(E(X))^2 = \bigcup_{\alpha\in(0,1]} \alpha\cdot[E(a(w,\alpha)),\ E(b(w,\alpha))]^2$$

$$= \bigcup_{\alpha\in(0,1]} \alpha[a'(\alpha),\quad b'(\alpha)]$$

according to L.A.Zadeh s extension principle. Where

$$a'(\alpha) = \min\{E^2(a(w,\alpha)),\ E(a(w,\alpha))\cdot E(b(w,\alpha),\ E^2(b(w,\alpha))\}$$

$$= \max\{E^2(a(w,\alpha)),\ E(a(w,\alpha))\cdot E(b(w,\alpha),\ E^2(b(w,\alpha))\}\ .$$

In the similar way, we have

$$(X^2(w))_\alpha = (X_\alpha(w))^2 = [\tilde{a}(w,\alpha),\ \tilde{b}(w,\alpha)]$$

where

$$\tilde{a}(w,\alpha) = \min\{a^2(w,\alpha),\ a(w,\alpha)\cdot b(w,\alpha),\ b^2(w,\alpha)\}$$

$$\tilde{b}(w,\alpha) = \max\{a^2(w,\alpha),\ a(w,\alpha)\cdot b(w,\alpha),\ b^2(w,\alpha)\}$$

then

$$E((X_\alpha)^2) = E((X^2)_\alpha)$$

$$= [E(\tilde{a}(w,\alpha)),\quad E(b(w,\alpha))]$$

So we have

$$D^2(X) = \bigcup_{\alpha\in(0,1]} \alpha\cdot[E(\tilde{a}(w,\alpha)) - b'(\alpha),\quad E(\tilde{b}(w,\alpha)) - a'(\alpha)]$$

Note: In general case, the above-proved theorem is incorrect.

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