BAYES PRINCIPLE AND FUZZY DISJOINTNESS

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ABSTRACT

The Bayes principle is connected with the notion of disjointness. In so far published papers devoted to the Bayes principle for fuzzy probability measures (see e.g. [3, 4, 6]), an apriori definition of the fuzzy disjointness is proposed and then the Bayes principle is studied. So in [3, 4] is defined the W-disjointness of two fuzzy subsets μ, ν by fulfilling of the inequality $\mu \leq 1 - \nu$. Similarly, in [6] is defined the F-disjointness of μ, ν by fulfilling of the inequality $\min\{\mu, \nu\} \leq 1/2$.

In the present paper we use reversal approach. Having given a fuzzy probability measure m, we study a family K_m consisting of all fuzzy set systems for which the Bayes formula holds, firstly. Our result leads to a new natural definition of fuzzy disjointness, namely to the m-disjointness. Two fuzzy subsets μ, ν are m-disjoint iff $m(\mu \wedge \nu) = 0$. The paper generalize the results of [3, 4, 6].

1. BAYES PRINCIPLE IN THE CRISP CASE

Let $(\Omega, \mathcal{A}, P)$ be a crisp probability space. Let $B \in \mathcal{A}$ be a crisp subset of Ω . Then the conditional probability given B is defined as follows:

$$\forall A \in \mathcal{A} : P(A/B) = \frac{P(A \cap B)}{P(B)} \qquad \text{for } P(B) > 0 ,$$

$$P(A/B) = 0 \qquad \text{for } P(B) = 0 \quad .$$

Let $A_i \in \mathcal{L}$, $i = 1,\dots, k$, be crisp subsets of Ω. Any system $\mathcal{A} = \{A_1, \dots, A_k\}$ induces a partition $\mathcal{B}_{\mathcal{A}}$ of the space Ω, $\mathcal{B}_{\mathcal{A}} = \{B_{\varkappa}, \varkappa \in \{-1, 1\}^k\}$, where

$$B_{\varkappa} = \bigcap_{i=1}^{k} A_i^{\varkappa(i)} \quad , \quad A_i^1 = A_i \; , \; A_i^{-1} = A_i' \text{ (complement of } A_i \text{)}.$$

Clearly, some of the subsets $B_{\varkappa}$ can be empty. The subsets $B_{\varkappa}$ of the system $\mathcal{B}_{\mathcal{A}}$ are pairwise disjoint. Let $\beta \in \{-1, 1\}^k$ and $C \in \mathcal{L}$, $P(C) > 0$, be given. Then the Bayes formula has the following form:

$$P(B_{\beta}/C) = \frac{P(B_{\beta})P(C/B_{\beta})}{\sum_{\mathcal{B}_{\mathcal{A}}} P(B_{\varkappa})P(C/B_{\varkappa})} \quad . \tag{1}$$

The Bayes formula (1) is equivalent to the formula for total probability, i.e. for any $C \in \mathcal{L}$

$$P(C) = \sum_{\mathcal{B}_{\mathcal{A}}} P(B_{\varkappa} \cap C) \quad . \tag{2}$$

It can be easily proved by induction that (2) is satisfied for arbitrary system $\mathcal{A}$ iff (2) is satisfied for arbitrary one-element system $\mathcal{A}$, i.e.

$$\forall C, A \in \mathcal{L} : P(C) = P(C \cap A) + P(C \cap A') \quad . \tag{3}$$

Note that $\mathcal{B}_{\{A\}} = \{A, A'\}$. So in the crisp case we have the following evident assertion:

A system $\mathcal{A} = \{A_1, \dots, A_k\}$ fulfilles the Bayes principle (i.e. (1) or (2)) iff $A_i \in K_p$, $i = 1, \dots, k$,

$$K_p = \{A \in \mathcal{L} \, , \; \forall C \in \mathcal{L} : P(C) = P(C \cap A) + P(C \cap A')\} \; .$$

In the crisp case we have $K_p = \mathcal{L}$.

2. BAYES PRINCIPLE IN THE FUZZY CASE

Now let (Ω, σ, m) be a fuzzy probability space in sense of Klement et al. [2] . We modify the ideas of the first part of this paper for the fuzzy case. Recall that a fuzzy probability measure on a fuzzy σ-algebra σ is in [2] defined as a continuous from below mapping $m:\sigma \longrightarrow [0\,,\,1]$ fulfilling two next properties:

$$m(\,0_{\Omega}\,) = 0 \qquad \text{and} \qquad m(\,1_{\Omega}\,) = 1 \tag{4}$$

$$\forall \mu, \gamma \in \sigma \;:\; m(\mu \vee \gamma\,) + m(\mu \wedge \gamma\,) = m(\mu) + m(\gamma\,). \tag{4'}$$

Let $\gamma \in \sigma$ be a fuzzy subset of Ω. The conditional fuzzy probability given γ is defined as follows (see e.g. [3]):

$$\forall \mu \in \sigma : c(\mu/\gamma\,,m) = \frac{m(\mu \wedge \gamma)}{m(\gamma\,)} \qquad \text{for } m(\gamma\,) > 0\,,$$

$$c(\mu/\gamma\,,m) = 0 \qquad \text{for } m(\gamma\,) = 0\,.$$

Any given system $\mathcal{A} = \{\mu_1, \ldots, \mu_k\} \in \sigma^k$, $k \in N$, of fuzzy subsets induces a fuzzy partition $\mathcal{B}_{\mathcal{A}}$ of Ω,

$$\mathcal{B}_{\mathcal{A}} = \{\,\gamma_{\chi}, \chi \in \{-1,\; 1\}^k\,\} \quad , \text{ where}$$

$$\gamma_{\chi} = \bigwedge_{i=1}^{k} \mu_i^{\chi(i)} \qquad , \mu^1 = \mu,\; \mu^{-1} = \mu' \text{ (fuzzy com-}$$

plement of μ) .

The validity of the fuzzy Bayes formula

$$\forall \eta \in \sigma\,,\; m(\eta\,) > 0,\; \beta \in \{-1,\; 1\}^k \;:$$

$$c(\gamma_{\beta}\,/\eta\,,m) \;=\; \frac{m(\gamma_{\beta}\,)c(\eta\,/\gamma_{\beta}\,,m)}{\sum\limits_{\mathcal{B}_{\mathcal{A}}} m(\gamma_{\chi}\,)c(\eta\,/\gamma_{\chi}\,,m)} \tag{5}$$

is equivalent to the validity of the next formula:

$$\forall \eta \in \sigma : \;\; m(\eta\,) = \sum_{\mathcal{B}_{\mathcal{A}}} m(\,\eta \wedge \gamma_{\chi}) \quad . \tag{6}$$

Denote $\quad K_m = \{\mu \in \sigma, \forall \eta \in \sigma : m(\eta\,) = m(\eta \wedge \mu) + m(\eta \wedge \mu')\}$.

<u>Lemma 1.</u> Let a system $\mathcal{A} = \{\mu_1, \ldots, \mu_k\} \in K_m^k$, $k \in N$, be given. Then formulas (5) and (6) hold.

<u>Proof.</u> It is enough to prove (6). Let $\eta \in \sigma$ be given. If $k = 1$, i.e. $\mathcal{A} = \{\mu_1\}$, then $\mathcal{B}_{\mathcal{A}} = \{\mu_1, \mu_1'\}$ and the definition of K_m

implies the validity of (6) . If $k>1$, we proceed by induction. Suppose that for all systems of K_m^{k-1} the formula (6) is true. Denote $\mathcal{A}^*=\{\mu_1,\dots,\mu_{k-1}\}$. Then $\forall \varkappa \in \mathcal{B}_{\mathcal{A}}: \nu_{\varkappa}=\nu_{\varkappa^*}\wedge\mu_k^{\varkappa(k)}$, $\varkappa\in\{-1,1\}^k$, $\varkappa(i)=\varkappa^*(i)$, $i=1,\dots,k-1$, $\nu_{\varkappa^*}\in\mathcal{B}_{\mathcal{A}^*}$.

The definition of K_m implies

$$m(\eta)=m(\eta\wedge\mu_k)+m(\eta\wedge\mu_k') .$$

On the other hand,

$$m(\eta\wedge\mu_k)=\sum_{\mathcal{B}_{\mathcal{A}^*}} m(\eta\wedge\mu_k\wedge\nu_{\varkappa^*}) ,$$

$$m(\eta\wedge\mu_k')=\sum_{\mathcal{B}_{\mathcal{A}^*}} m(\eta\wedge\mu_k'\wedge\nu_{\varkappa^*}) .$$

It follows

$$m(\eta)=\sum_{\mathcal{B}_{\mathcal{A}^*}}(m(\eta\wedge\nu_{\varkappa^*}\wedge\mu_k)+m(\eta\wedge\nu_{\varkappa^*}\wedge\mu_k'))=\sum_{\mathcal{B}_{\mathcal{A}}} m(\eta\wedge\nu_{\varkappa}) .$$

<u>Lemma 2.</u> Let $\mathcal{A}=\{\mu_1,\dots,\mu_k\}\in\sigma^k$, $k\in N$, be a given system of fuzzy subsets and let (5) or (6) hold. Then $\mu_i\in K_m$, $i=1,\dots,k$. Moreover, for any $\varkappa\in\{-1,1\}^k$, $\nu_{\varkappa}\in K_m$.

<u>Proof.</u> We can suppose that (6) holds. It is enough to prove $\mu_k\in K_m$. For $k=1$ is this assertion evident. Let $\eta\in\sigma$, $k>1$. Then (6) implies

$$m(\eta)=\sum_{\mathcal{B}_{\mathcal{A}}} m(\eta\wedge\nu_{\varkappa})$$

$$m(\eta\wedge\mu_k)=\sum_{\mathcal{B}_{\mathcal{A}}} m(\eta\wedge\nu_{\varkappa}\wedge\mu_k)=\sum_{\mathcal{B}_{\mathcal{A}^*}}(m(\eta\wedge\nu_{\varkappa^*}\wedge\mu_k)+m(\eta\wedge\nu_{\varkappa^*}\wedge\mu_k'\wedge\mu_k)),$$

$$m(\eta\wedge\mu_k')=\sum_{\mathcal{B}_{\mathcal{A}}} m(\eta\wedge\nu_{\varkappa}\wedge\mu_k')=\sum_{\mathcal{B}_{\mathcal{A}^*}}(m(\eta\wedge\nu_{\varkappa^*}\wedge\mu_k')+m(\eta\wedge\nu_{\varkappa^*}\wedge\mu_k\wedge\mu_k')).$$

It follows

$$m(\eta\wedge\mu_k)+m(\eta\wedge\mu_k')=m(\eta)+m(\eta\wedge\mu_k\wedge\mu_k') . \tag{7}$$

Now, take $\nu_\beta\in\mathcal{B}_{\mathcal{A}}$. By (6) we have

$$m(\nu_\beta)=\sum_{\mathcal{B}_{\mathcal{A}}} m(\nu_\beta\wedge\nu_{\varkappa})=m(\nu_\beta)+\sum_{\varkappa\neq\beta} m(\nu_\beta\wedge\nu_{\varkappa}) \quad , \text{ so that}$$

$$m(\nu_\beta\wedge\nu_{\varkappa})=0 \quad , \beta\neq\varkappa \tag{8}$$

For any $\varkappa$ denote $\varkappa'$ the k-tuple differing from $\varkappa$ only in the last coordinate. Then $\varkappa \neq \varkappa'$ and (8) implies

$$m(\gamma_{\varkappa} \wedge \mu_k \wedge \mu_k') = m(\gamma_{\varkappa} \wedge \gamma_{\varkappa'}) = 0 \ .$$

Then

$$m(\mu_k \wedge \mu_k') = \sum m(\gamma_{\varkappa} \wedge \mu_k \wedge \mu_k') = 0 \ . \tag{9}$$

(9) together with (7) implies

$$m(\eta) = m(\eta \wedge \mu_k) + m(\eta \wedge \mu_k') \ ,$$

so that $\mu_k \in K_m$.

Now, let $\beta \in \{-1, 1\}^k$ be given. We have

$$(\gamma_\beta)' \wedge \gamma_\varkappa = \gamma_\varkappa \quad \text{for } \varkappa \neq \beta \ , \text{ and} (\gamma_\beta)' \wedge \gamma_\beta = \bigvee_{\varkappa \neq \beta} (\gamma_\varkappa \wedge \gamma_\beta) \ ,$$

what implies (together with (8))

$$m((\gamma_\beta)' \wedge \gamma_\beta) \leq \sum_{\varkappa \neq \beta} m(\gamma_\varkappa \wedge \gamma_\beta) = 0 \ .$$

The validity of (6) leads to the following equality:

$$\forall \eta \in \mathfrak{S} : m(\eta) = \sum_{\mathcal{B}_A} m(\eta \wedge \gamma_\varkappa) = m(\eta \wedge \gamma_\beta) + \sum_{\varkappa \neq \beta} m(\eta \wedge \gamma_\varkappa) =$$

$$= m(\eta \wedge \gamma_\beta) + \sum_{\varkappa \neq \beta} m(\eta \wedge (\gamma_\beta)' \wedge \gamma_\varkappa) + m(\eta \wedge (\gamma_\beta)' \wedge \gamma_\beta) =$$

$$= m(\eta \wedge \gamma_\beta) + m(\eta \wedge (\gamma_\beta)') \ .$$

So we have $\gamma_\beta \in K_m$.

Lemmas 1, 2 imply that K_m is a system of all fuzzy subsets of $\mathfrak{S}$ for which the Bayes principle (when the fuzzy probability measure m is taken into account) remains valid. The next proposition gives some other properties of the system K_m.

Proposition 1. Let m be a fuzzy probability measure in sense of Klement et al. Then:

i) $\mu \in K_m$ iff $m(\mu \vee \mu') = 1$ and $m(\mu \wedge \mu') = 0$;

ii) K_m is a soft fuzzy algebra, i.e. a fuzzy algebra not containing the fuzzy subset $(1/2)_\Omega$;

iii) if $\tau \in \sigma$ is a sharpening of some $\mu \in K_m$, i.e. if

$$|\tau - 1/2| \geq |\mu - 1/2| \quad \text{then } \tau \in K_m .$$

<u>Proof.</u> i) Let $\mu \in K_m$. Then (6) implies

$$m(\mu) = m(\mu \wedge \mu) + m(\mu \wedge \mu') = m(\mu) + m(\mu \wedge \mu') , \text{ so that}$$

$$m(\mu \wedge \mu') = 0 .$$

Further,

$$m(\mu \vee \mu') = m((\mu \vee \mu') \wedge \mu) + m((\mu \vee \mu') \wedge \mu') =$$

$$= m(\mu) + m(\mu') = m(1_\Omega \wedge \mu) + m(1_\Omega \wedge \mu') = m(1_\Omega) = 1 .$$

Note that m on K_m has the complementation property

$$m(\mu') = 1 - m(\mu) .$$

Conversely, let $\mu \in \sigma$, $m(\mu \wedge \mu') = 0$, $m(\mu \vee \mu') = 1$. Then for any $\eta \in \sigma$ we have (see e.g. [5])

$$m(\eta) = m(\eta \wedge (\mu \vee \mu')) = m((\eta \wedge \mu) \vee (\eta \wedge \mu')) \quad \text{and}$$

$$m(\eta \wedge (\mu \wedge \mu')) = m((\mu \wedge \eta) \wedge (\eta \wedge \mu')) \leq m(\mu \wedge \mu') = 0 .$$

The valuation property (4') then implies

$$m(\eta) = m((\mu \wedge \eta) \vee (\eta \wedge \mu')) + m((\eta \wedge \mu) \wedge (\eta \wedge \mu')) =$$

$$= m(\eta \wedge \mu) + m(\eta \wedge \mu') , \text{ i.e. } \mu \in K_m .$$

ii) Let $\mu, \nu \in K_m$. According to i) it is evident that $\mu', \nu' \in K_m$.

Further,

$$m((\mu \wedge \nu) \wedge (\mu \wedge \nu)') = m((\mu \wedge \nu \wedge \mu') \vee (\mu \wedge \nu \wedge \nu')) \leq$$

$$\leq m(\mu \wedge \mu') + m(\nu \wedge \nu') = 0$$

and

$$m((\mu \wedge \nu) \vee (\mu \wedge \nu)') = m((\mu \vee \mu' \vee \nu') \wedge (\nu \vee \mu' \vee \nu')) =$$

$$= m(\mu \vee \mu' \vee \nu') + m(\nu \vee \mu' \vee \nu') - m(\mu \vee \nu \vee \mu' \vee \nu') = 1 ,$$

so that $\mu \wedge \nu \in K_m$.

Thus, the system K_m is a fuzzy algebra. Let $(1/2)_\Omega \in K_m$. Then i) implies $m((1/2)_\Omega) = 1$ and $m((1/2)_\Omega) = 0$ what is a contradiction. We get that K_m is a soft fuzzy algebra.

iii) If τ is a sharpening of a $\mu \in K_m$, then $\tau \vee \tau' \geq \mu \vee \mu'$ and $\tau \wedge \tau' \leq \mu \wedge \mu'$ so that

$m(\tau \vee \tau') \geq m(\mu \vee \mu') = 1$ and $m(\tau \wedge \tau') \leq m(\mu \wedge \mu') = 0$.

It follows $\tau \in K_m$.

3. FUZZY DISJOINTNESS

Let $\mathcal{A} \in K_m^k$, $k \in N$ be given and let $\mathcal{B}_{\mathcal{A}}$ be the corresponding fuzzy partition. As it is stated in part 2., the Bayes principle, actually (5) and (6), is true. However, $\mathcal{B}_{\mathcal{A}}$ can include some fuzzy subsets γ_χ of measure zero, i.e. $m(\gamma_\chi) = 0$. Excluding such elements we obtain a system $\mathcal{E}_{\mathcal{A}} = \{\gamma_\chi \in \mathcal{B}_{\mathcal{A}}, m(\gamma_\chi) > 0\}$. For this new system (5) and (6), i.e. the Bayes principle, remains valid. System $\mathcal{E}_{\mathcal{A}}$ has the properties:

(C1) $\forall \mu, \gamma \in \mathcal{E}_{\mathcal{A}}, \mu \neq \gamma : m(\mu \wedge \gamma) = 0$;

(C2) $m(\sup\limits_{\mathcal{E}_{\mathcal{A}}} \gamma) = 1$;

(C3) $\forall \gamma \in \mathcal{E}_{\mathcal{A}} : m(\gamma) > 0$.

(C2) and (C3) are identical to (R2) and (R3) of the Piasecki's definition of a W-fuzzy Bayes partition (see e.g. [3]). Note that (R2) and (R3) are preserved in the definition of a F-fuzzy Bayes partition (see [6]) also. (R1), respectively (R1') property of mutual W-disjointness (F-disjointness) of a Bayes fuzzy partition corresponds to our (C1). Thus, one natural definition of disjointness of two fuzzy subsets can be as follows:

<u>Definition.</u> Let (Ω, σ, m) be a fuzzy probability space in sense of Klement et al., $\mu, \gamma \in \sigma$. Then μ, γ are m-disjoint iff $m(\mu \wedge \gamma) = 0$.

<u>Remark.</u> Note that the m-disjointness on K_m is more general than the W-disjointness and the F-disjointness, respectively.

That means if $\mu, \nu \in K_m$ are W-(F-)disjoint, then they are m-disjoint. The reverse assertion may fail, see Example 1, part iii).

It is easy to see that m on K_m fulfils two following properties:

(D1) for any $\mu \in K_m$ we have

$m(\mu \vee \mu') = 1$;

(D2) for any sequence $\{\mu_k\} \in K_m^N$, $\sup\{\mu_k\} \in K_m$, satisfying the condition (C1) we have

$m(\sup\{\mu_k\}) = \sum m(\mu_k)$.

(D1) is identical to (P1) of the Piasecki's definition of a fuzzy P-measure (see e.g. [3, 5]). (D2) corresponds to (P2) property of a fuzzy P-measure replacing the W-disjointness by the m-disjointness. The above remark implies that m is a fuzzy P-measure on K_m.

Our results are summarized in the following general version of Piasecki's theorem on the Bayes formula for fuzzy probability measures.

<u>Proposition 2.</u> Let (Ω, σ, m) be a fuzzy probability space in sense of Klement et al., K_m be a system consisting of all fuzzy partitions of (Ω, σ) satisfying the Bayes formula on (Ω, σ, m). Then m is a fuzzy P-measure on the soft fuzzy algebra K_m.

<u>Example 1.</u> Let Ω be the unit interval $[0 , 1]$, $\sigma = \mathbb{F}(\mathcal{B})$ be a generated fuzzy σ-algebra of all Borel-measurable fuzzy subsets of Ω, and λ be the Lebesque measure. Then

$m(\mu) = \lambda(\mu > c)$, where $c \in]0 , 1[$

is a fuzzy probability space in sense of Klement et al. Then

i) if $c < 1/2$, then

$K_m = \{ \mu \in \sigma ,\ \lambda(c < \mu < 1 - c) = 0 \}$,

K_m is a soft fuzzy σ-algebra ;

ii) if $c \geq 1/2$, then

$K_m = \{ \mu \in \sigma ,\ \lambda(1 - c \leq \mu \leq c) = 0 \}$,

K_m is a soft fuzzy algebra but it is not a fuzzy σ-algebra ;

iii) let $\mu = 1_\Omega$, $\nu = \begin{cases} 1 & \text{for rational points} \\ 0 & \text{for irational points} \end{cases}$;

then μ and ν are neither W-disjoint nor F-disjoint but for every $c \in]0, 1[$, μ, ν belong to K_m and $m(\mu \wedge \nu) = 0$ so that μ, ν are m-disjoint.

Example 2. Let $(\Omega, \mathcal{A}, P)$ be a crisp probability space, $\sigma = \mathbb{F}(\mathcal{A})$ be a generated fuzzy σ-algebra, $f, g \in \sigma$, $\forall x \in \Omega$, $f(x) < g(x)$. We define a Markoff kernel K (for more details see e.g. [1]) letting $K(x,.)$ be an uniform distribution on the interval $[f(x), g(x)]$ for $x \in \Omega$.

The relationship

$$m(\mu) = \int_\Omega K(x, [0, \mu(x)[\,)dP(x) \quad , \mu \in \sigma$$

defines a fuzzy probability measure in sense of Klement et al. (see [1]). Then

$$K_m = \{ \mu \in \sigma ,\ P(1/2 - h(x) < \mu < 1/2 + h(x)) = 0 \quad ,$$

where $h(x) = \max\{|1/2 - f(x)|, |1/2 - g(x)|\}$.

If $f = 0_\Omega$ and $g = 1_\Omega$, then m is the Zadeh's fuzzy probability measure [7] and $K_m = \mathcal{A}$, i.e. only the crisp partitions fulfils the Bayes principle.

REFERENCES

[1] E.P.Klement, Characterization of finite fuzzy measures using Markoff-kernels, J.Math.Anal.Appl.75 (1980) 330-339.

[2] E.P.Klement, R.Lowen and W.Schwychla, Fuzzy probability measures, Fuzzy Sets and Systems 5 (1981) 21-30 .

[3] K.Piasecki, On the Bayes formula for fuzzy probability measures, Fuzzy Sets and Systems 18 (1986) 183-185 .

[4] K.Piasecki, Note to [3] , Fuzzy Sets and Systems 24 (1987) 121-122 .

[5] K.Piasecki, Fuzzy P-measures and their application in decision makong, in: M.Fedrizzi, j.Kacprzyk (eds.), Combining fuzzy imprecision with probabilistic uncertainty in decision making, Springer Ver., 1987 .

[6] K.Piasecki and Z.Świtalski, A remark on the definition of fuzzy P-measures and the Bayes formula, Fuzzy Sets and Systems 26 (1988) .

[7] L.A.Zadeh, Probability measures of fuzzy events, J.Math.Anal.Appl 23 (1968) 421-427 .