THE HAHN-JORDAN DECOMPOSITION ON FUZZY QUANTUM SPACES

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In the present note, we give a generalization of the Hahn-Jordan decomposition of signed measures on so-called fuzzy quantum spaces, as well as the Lebesgue decomposition theorem, is present.

The following definition has been introduced in [3,6]:

DEFINITION 1. A fuzzy quantum space is a couple (X,M), where X is a nonempty set and $M \subset [0,1]^X$ such that the following conditions are satisfied:

(i) if $[1]_X(x) = 1$ for any $x \in X$, then $[1]_X \in M$;

(ii) if $a \in M$, then $a^{\perp} := 1 - a \in M$;

(iii) if $[1/2]_X(x) = 1/2$ for any $x \in X$, then $[1/2]_X \notin M$;

(iv) $\bigcup_{n=1}^{\infty} a_n := \sup_n a_n \in M$, for any $\{a_n\}_{n=1}^{\infty} \subset M$.

The system M is called in the fuzzy sets theory a soft fuzzy σ-algebra (Piasecki, K. [4]).

Using Piasecki [4], we define a P-measure for a fuzzy quantum space as follows:

DEFINITION 2. A P-measure is any mapping $m: M \to [0,1]$, such that

(i) $m(a \cup a^{\perp}) = 1$ for any $a \in M$; (1.1)

(ii) $m(\bigcup_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} m(a_n)$ whenever $\{a_n\}_{n=1}^{\infty} \subset M$, (1.2)

$a_i \leq 1 - a_j$ for $i \neq j$.

Due to Piasecki [4], we say that a fuzzy subset $a \in M$ is a W-empty set (W-universum) if $a \leq a^{\perp}$ ($a^{\perp} \leq a$) and we denote by $W(M)$ the set of all fuzzy W-empty sets from M. Two sets a and b of M are said orthogonal and we write $a \perp b$ (W-separated, in terminology of [4]) if $a \leq b^{\perp}$.

The properties (1.1) and (1.2) motivate us to define a signed measure for a fuzzy quantum space (X,M) as follows:

DEFINITION 3. Let M be a soft fuzzy σ-algebra of subsets of a set X. A mapping $m: M \to R$ such that

(i) $m(a \cup a^{\perp}) = m([1]_X)$ for any $a \in M$;

(ii) $m(\bigcup_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} m(a_n)$ if $\{a_n\} \subset M$, $a_i \leq a_j^{\perp}$, $i \neq j$,

is said to be <u>a signed measure</u> of (X,M).

If $m(a) \geq 0$ for any $a \in M$, m is said to be a measure, in particular, if for a measure m we have $m([1]_X) = 1$, m is a P-measure.

By Dvurečenskij [1], for a signed measure on M the following properties hold:

THEOREM 1. Let m be a signed measure on M. Then

(i) $m(a^{\perp}) = m(1) - m(a)$ for any $a \in M$;

(ii) $m(x) = 0$ for any $x \in W(M)$;

(iii) $m(b) = m(a) + m(b \cap a^{\perp})$ if $a \leq b$, $a,b \in M$;

(iv) $m(a) = m(a \cap x)$ for any $a \in M$ and any $x^{\perp} \in W(M)$;

(v) $m(a \cup y) = m(a)$ for any $a \in M$ and any $y \in W(M)$;

(vi) $m(a \cup b) + m(a \cap b) = m(a) + m(b)$ for all $a,b \in M$;

(vii) if $a_n \nearrow a$ ($a_n \searrow a$), then $m(a_n) \to m(a_n)$, $\{a_n\} \subset M$, $a \in M$.

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THEOREM 2. Let m be a signed measure on M, then every system of mutually orthogonal sets $a \in M$, $m(a) > 0$ ($m(a) < 0$) is countable.

PROOF. Let $\mathcal{E} \subset M$ be a system of mutually orthogonal sets $a \in M$ with $m(a) > 0$.
For n = 1, 2, ..., let $\mathcal{E}_n = \{a: m(a) > 1/n\}$, then

$$\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n. \qquad (1.3)$$

It is clear that the system $\mathcal{E}_n$ is finite for every n and from the property (1.3) we imply that the system $\mathcal{E}$ is countable.

Q.E.D.

THEOREM 3. Let $a \in M$ and let $|m(a)| < \infty$, then every system $\mathcal{F}$ of mutually orthogonal sets b with $b \leq a$ and $m(b) > 0$ ($m(b) < 0$) is countable.

PROOF. It is analogous as that of Theorem 2.

DEFINITION 4. The fuzzy set $a \in M$ is <u>positive</u> (<u>negative</u>) with respect to a signed measure m if, for every set $b \in M$,

$m(a \cap b) \geq 0$ ($m(a \cap b) \leq 0$).

DEFINITION 5. A couple (a,b), where a is positive and b is negative set with respect to m such that $b = a^{\perp}$, is called the <u>Hahn decomposition</u> of (X,M) with respect to a signed measure m.

THEOREM 4. (<u>Hahn decomposition</u>) Let m be a signed measure on M, then the Hahn decomposition of (X,M) with respect to m exists.

PROOF. Without loss of generality we may assume that there is a maximal system of mutually orthogonal sets $a \in M$, which are negative ($m(a) < 0$) with respect to m (In opposite case m is a measure and we put $a = [1]_X$, $b = [0]_X$.). By Theorem 1, the system $\mathcal{E}$ is countable.

Let $b = \bigcup \{a: a \in \mathcal{E}\} \in M$ and let $a = b^{\perp} \in M$, then

$m(b) = m(\bigcup a) = \sum m(a) < 0.$

For every $c \in M$, one holds

$m(c \cap b) = m(c \cap \bigcup_i a_i) = m(\bigcup_i (c \cap a_i)) = \sum_i m(c \cap a_i) \leq 0$, that is, b is negative.

Now we show that a is positive. Let a be not positive, then there exists a $c_o \in M$, such that $c_o \leq a$, $m(c_o) < 0$. We denote by $\mathcal{E}_o$ a maximal system of mutually orthogonal sets $d \in M$, $d \leq c_o$, $m(d) > 0$. In view of Theorem 3, $\mathcal{E}_o$ is countable.

Let $d_o = \bigcup \{d: d \in \mathcal{E}_o\}$, then $m(d_o) > 0$, $d_o \leq c_o$. We show, that $c_o \cap d_o^{\perp}$ is negative, because it does not contain any set of a positive measure. From the equality (iii) of Theorem 1, $m(c_o) = m(c_o \cap d_o^{\perp}) + m(d_o)$ which entails $m(c_o \cap d_o^{\perp}) < 0$. Then the set $c_o \cap d_o^{\perp}$ is negative and $(c_o \cap d_o^{\perp}) \perp b$, which is a contradiction with the maximality of system $\mathcal{E}$.

Q.E.D.

THEOREM 5. Let (a_1,b_1) and (a_2,b_2) be two Hahn decompositions of (X,M) with respect to m, then

$$m(x\cap a_1) = m(x\cap a_2),$$
$$m(x\cap b_1) = m(x\cap b_2)$$

for any $x\in M$.

PROOF. Since $x\cap(a_1\cap a_2^{\perp})\leq x\cap a_1\leq a_1$, then

$$m(x\cap(a_1\cap a_2^{\perp}))\geq 0, \tag{1.4}$$

analogicaly, $x\cap(a_1\cap a_2^{\perp})\leq x\cap b_2$, then

$$m(x\cap(a_1\cap a_2^{\perp}))\leq 0. \tag{1.5}$$

Due to (1.4) and (1.5), it holds

$$m(x\cap(a_1\cap a_2^{\perp})) = 0. \tag{1.6}$$

Analogicaly we prove

$$m(x\cap(a_2\cap a_1^{\perp})) = 0. \tag{1.7}$$

From (1.6) and (1.7) we imply $m(x\cap a_1) = m(x\cap a_1\cap a_2) = m(x\cap a_2)$.

Analogicaly we prove $m(x\cap b_1) = m(x\cap b_2)$.

Q.E.D.

THEOREM 6. (Jordan decomposition) Let m be a signed measure on M and let (a,b) be any Hahn decomposition with respect to m. Then a mapping m^+ and m^- defined via

$$m^+(x) = m(x\cap a) \tag{1.8}$$
$$m^-(x) = m(x\cap b) \tag{1.9}$$

for any $x\in M$ are the measures on M and m^+, m^- are independent of given Hahn decomposition. Moreover, for any $x\in M$, it holds

$$m(x) = m^+(x) - m^-(x). \tag{1.10}$$

PROOF. It follows immediately from Theorem 5 and from the definition of the Hahn decomposition.

Q.E.D.

DEFINITION 6. The formula (1.10) is called a Jordan decomposition of a signed measure m. The measures m^+ and m^- are said to be positive and negative parts of m. The measure $|m|$ defined as

$$|m| = m^+ + m^- \tag{1.11}$$

is called a total variation of a signed measure m.

For some properties of the Jordan decomposition we recall the following notions.

An F-observable on a fuzzy quantum space (X,M) is a mapping $x: B(R^1) \longrightarrow M$ satisfying the following properties:

(i) $x(E^c) = 1 - x(E)$ for every $E \in B(R^1)$;

(ii) if $\{E_n\}_{n=1}^{\infty} \subset B(R^1)$, then $x(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} x(E_n)$,

where $B(R^1)$ is the Borel σ-algebra of the real line R^1, and E^c denotes the complement of E in R^1.

If $f: R^1 \longrightarrow R^1$ is a Borel measurable function and x is an F-observable, then $f \circ x: E \longrightarrow x(f^{-1}(E))$, $E \in B(R^1)$, is an F-observable, too.

The product of two observables x and y is defined as follows

$$x.y = ((x + y)^2 - x^2 - y^2)/2. \tag{1.12}$$

If m is a P-measure and x is an observable of (X,M), then $m: E \longrightarrow m(x(E))$, $E \in B(R^1)$. Let a be given set of M, the indicator of a is a unique observable x_a defined via

$$x_a(E) = \begin{cases} a \cap a^{\perp} & 0,1 \notin E \\ a^{\perp} & 0 \in E,\ 1 \notin E \\ a & 0 \notin E,\ 1 \in E \\ a \cup a^{\perp} & 0,1 \in E, \end{cases} \qquad (1.13)$$

for $E \in B(R^1)$.

A mean value of an observable x in a P-measure m we understand the expression $m(x) := \int x\, dm$ defined by

$$\int x\, dm = \int_{R^1} t\, dm_x(t)$$

(if the integral on the right hand exists and is finite).

Let m be a P-measure of (X,M). We define an indefinite integral of an observable x over a fuzzy set $a \in M$ via

$$m(a) = \int_a x\, dm := \int x . x_a\, dm,$$

where x_a is the indicator of a fuzzy set.
This indefinite integral has been defined in [2], and in another way in [5].

THEOREM 7. Let m be a P-measure and let x be an observable of (X,M) such that $m(|x|) := \int_{R^1} |t|\, dm_x(t) < \infty$. Then the mapping ν defined via

$$\nu(a) = \int_a x\, dm, \quad a \in M, \qquad (1.14)$$

is a signed measure on (X,M), where negative and positive parts, ν^+ and ν^-, are defined as follows

$$\nu^+(a) = \int_a x^+\, dm,\ a \in M,$$
$$\nu^-(a) = \int_a x^-\, dm,\ a \in M, \qquad (1.15)$$

where $x^+ = f^+ \circ x$, $x^- = f^- \circ x$ and $f^+(t) = \max(t,0)$, $f^-(t) = - \min(0,t)$, $t \in R^1$.

PROOF. In this proof, we present two qualitatively different approaches to it.

PROOF 1. Let $x^+ = f^+_\circ x$, $x^- = f^-_\circ x$, then (a,b) is a Hahn decomposition of M defined via $a = x([0,\infty))$, $b = x((-\infty,0))$. Let $x^+(R^1) = x((f^+)^{-1}(R^1)) = x(R^1)$.

We can show that $m(e \cap a) \geqslant 0$, $m(e \cap b) \leqslant 0$ for any $e \in M$. Denote by $1_k = (a \cup a^\perp) \cap (e \cup e^\perp) \cap x(R^1)$ and we define new observables of (X,M):

$$\bar{x}(E) = x(E) \cap 1_k \cup 0_k$$

$$\bar{x}_a(E) = x_{\bar{a}}(E)$$

$$\bar{x}_e(E) = x_{\bar{e}}(E)$$

$$\overline{x_{a\cap e}}(E) = x_{\overline{a \cap e}}(E) \cap 1_k \cup 0_k, \quad E \in B(R^1), \text{ where}$$

$$\bar{a} = a \cap 1_k \cup 0_k$$

$$\bar{e} = e \cap 1_k \cup 0_k.$$

Then $\overline{x_{a\cap e}} = x_{\overline{a\cap e}} = x_{\bar{e}} \cdot x_{\bar{a}}$ and due to Dvurečenskij [1], there exists a mapping $\varphi : X_{1k} \longrightarrow R^1$ such that $x_{1k}(\bar{x}(E)) = \varphi^{-1}(E)$ for every $E \in B(R^1)$, where $X_{1k}(b) = \{ s \in X_{1k} : b(s) = 1_k(s)\}$, where b is a fuzzy set beloging to M such that $(b \cup b^\perp)(s) = 1_k(s)$ for any $s \in X$. The system $\mathcal{A}_{1k}$ of all fuzzy sets $X_{1k}(b)$ forms a σ-algebra of crisp subsets of a set $X_{1k}([1])$ and a mapping $\mu_{1k} : \mathcal{A}_{1k} \longrightarrow [0,1]$ defined via

$$\mu_{1k}(X_{1k}(b)) = m(b), \quad b \in \mathcal{A}_{1k}$$

is a probability measure.

It may be proved that

$$X_{1k}(\bar{x}_a(E)) = I_A^{-1}(E)$$

$$X_{1k}(\bar{x}_e(E)) = I_F^{-1}(E)$$

for any $E \in B(R^1)$, where $A = \{ s \in X_{1k} : \varphi(s) \geqslant 0\}$ and F is a (unique) crisp subset of $\mathcal{A}_{1k}$.

Calculate

$$\nu^+(a) = \nu(e\wedge a) = \int_{e\wedge a} x\, dm = \int_{A\cap F} \varphi(s)\, d\mu_{1k}(s) \geq 0.$$

Analogically we have

$$\nu^-(a) = -\nu(e\wedge b) = -\int_{e\wedge a} x\, dm = -\int_{A^c\cap F} \varphi(s)\, d\mu_{1k}(s) \geq 0.$$

In same way we have

$$\nu^-(e) = -\int_e x^-\, dm,$$

which finishes the first proof.

PROOF 2. Let us define the equivalence "$\sim$" via $a \sim b$ iff $m(a\wedge b^{\perp}) = 0 = m(a^{\perp}\wedge b)$. Then $\widetilde{M} = M/\sim = \{\widetilde{a} = \{b\in M: b\sim a\}: a\in M\}$ is a Boolean σ-algebra and μ on M defined as

$$\mu(\widetilde{a}) = m(a),\ a\in M$$

is a probability measure on M.

The map $\widetilde{x}: E \longrightarrow \widetilde{x}(E)$, $E\in B(R^1)$, is a σ-homomorphism from $B(R^1)$ into M. Due to Sikorski [7], there is a measurable space $(\Omega, \mathcal{S})$ and a σ-homomorphism h from $\mathcal{S}$ onto M. In view of Varadarajan [8], there is an $\mathcal{S}$-measurable, real-valued function Ψ such that $\widetilde{x}(E) = h(\Psi^{-1}(E))$, $E\in B(R^1)$. It is simple to verify, that $P_m: \mathcal{S} \longrightarrow [0,1]$ which is given by $P_m(A) = \mu(h(A))$, $A\in\mathcal{S}$, is a probability measure on $\mathcal{S}$. Hence, for any $e\in M$, there is $G\in\mathcal{S}$ such that $\widetilde{e} = h(G)$. Moreover, if we put $A^+ = \{\omega\in\Omega: \Psi(\omega)\geq 0\}$ and $A^- = \{\omega\in\Omega: \Psi(\omega) < 0\}$, then

$$\widetilde{a} = h(A^+),\ \widetilde{b}^{\perp} = h(A^-).$$

Calculate

$$\nu^+(e) = \nu(e\wedge a) = \int_{e\wedge a} x\, dm = \int_{\widetilde{e}\wedge\widetilde{a}} \widetilde{x}\, d\mu = \int_{G\cap A^+} \Psi(\omega)\, dP_m(\omega) \geq 0.$$

$$\nu^-(e) = -\nu(e \cap b) = -\int_{e \cap b} x \, dm = \int_{G \cap A^-} \psi(\omega) \, dP_m(\omega) \geq 0.$$

Q.E.D.

To define a Lebesgue decomposition theorem we introduce the following notions.

DEFINITION 7. We say that a signed measure m is <u>dominated</u> by a measure n, if $n(a) = 0$ implies $m(a) = 0$, and we write $m \ll n$.

DEFINITION 8. If m, n are two measures on M, we say m is <u>singular</u> with respect to n (we write $m \perp n$) if there exist two fuzzy sets $a, b \in M$, $a = b^{\perp}$, such that for every $x \in M$, $m(x \cap a) = n(x \cap b) = 0$.

THEOREM 8. (<u>Lebesgue decomposition</u>) Let m and n be measures on a fuzzy quantum space (X,M). Then there exist unique two measures m_1, m_2 on M with $m_1 \ll n$, and $m_2 \perp n$, such that

$$m(a) = m_1(a) + m_2(a)$$

for every $a \in M$.

PROOF. If $m \ll n$, we denote $m_1 = n$ and $m_2 = 0$ and Theorem 8 holds. Hence, we may assume that m is not dominated by n. We denote by $\mathcal{E}$ a system of all sets $a \in M$, $m(a) = 0$, $n(a) > 0$.

Let $\mathcal{E}_o$ be a maximal system of mutually orthogonal sets from $\mathcal{E}$, thet $\mathcal{E}_o$ is countable (Theorem 7). Let $a_o = \bigcup\{a: a \in \mathcal{E}_o\} \subset M$, then $m_1(a) = m(a \cap a_o^{\perp})$ for every $a \in M$. If $a \in M$, $n(a) = 0$, then $n(a \cap a_o^{\perp}) = 0$ implies

$$m(a \cap a_o^{\perp}) = m_1(a) = 0. \qquad (1.16)$$

From (1.16) we have

$m_1 \ll n.$

Let, for every $a \in M$, we have $m_2(a) = m(a \cap a_o)$. Now we show that $m_2 \perp n$. Let $x = a_o$, $y = a_o^{\perp}$, then

$$0 \leq n(a \cap x) = n(a \cap a_o) \leq n(a_o) = 0, \tag{1.17}$$

and

$$m_2(a \cap y) = m(a \cap y \cap a_o) = m(a \cap a_o^{\perp} \cap a_o) = 0, \tag{1.18}$$

for every $a \in M$.

From (1.17) and (1.18) we conclude that $m_2 \perp n$.

Now we prove the uniqueness of m_1 and m_2. Let $n = m_1 + m_2 = m_1' + m_2'$, $m_1, m_1' \ll n$ and $m_2, m_2' \perp n$. Let a_1, b_1 and a_2, b_2 are such elements of M that $b_1 = a_1^{\perp}$, $b_2 = a_2^{\perp}$ and

$n(x \cap a_1) = m_2(x \cap b_1) = 0$

$n(x \cap a_2) = m_2'(x \cap b_2) = 0$ for any $x \in M$.

Calculate

$0 = n(x \cap a_1) = m_1(x \cap a_1) + m_2(x \cap a_1)$ which gives $m_1(x \cap a_1) = 0 = m_2(x \cap a_1)$.

Analogically $0 = n(x \cap a_2) = m_2'(x \cap a_2) = 0$

and

$n(x \cap a_1^{\perp}) = m_1(x \cap a_1^{\perp}) + m_2(x \cap a_1^{\perp}) = m_1(x \cap a_1^{\perp})$,

$m_1(x) = m_1(x \cap (a_1^{\perp} \cup a_1)) = m_1(x \cap a_1^{\perp}) + m_1(x \cap a_1) = m_1(x \cap a_1^{\perp})$.

Similary we show that

$n(x \cap a_2^{\perp}) = m_1'(x \cap a_2^{\perp}) = m_1'(x)$,

so that, $m_1(x) = m_1'(x)$ for any $x \in M$ and, therefore, $m_2(x) = m_2'(x)$ for each $x \in M$.

Q.E.D.

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