MULTISETS, FUZZY SETS AND COMPUTING

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1. Introduction

Petri nets are currently used to model concurrent computing situations. They employ "multisets" or "bags", which are essentially functions with values in the natural number system. Fuzzy sets, functions with values into the closed unit interval [0,1] are widely employed to model nondeterminism. Both of them of course are generalization of ordinary sets.

The objective of this paper is to point out that the theory of multisets and the theory of fuzzy sets are very close relatives. We do it by pointing out to a common framework namely K-subsets, which were proposed by Eilenberg. We also propose that it is more convenient and is of wider significance to replace K-subsets by R-sets, where R is an additive domain. The structure of an additive domain has been recently used by Manes to discuss assertion semantics of programming statements.

With the assistance of R-sets, we can present a generalized version of Petri nets and regard them as variants of K-Σ automata of Eilenberg. We do so and list certain affinities.

2. Definitions and Preliminaries

In this section some of the necessary definitions and concepts of Petri nets, automata and additive domains will be presented. We do it for the sake of completeness only.

Definition 1: A K-subset of X is a function $f:X \longrightarrow K$, where K is a semi-ring. If $K = \mathbb{N}$, then this function is known as multiset.

Definition 2: A Petri net is a 4-tuple $N = (B,E,F,M_0)$, where B is a non-empty set of conditions, E is a set of events, F is a multiset of $(B\times E) \cup (E\times B)$, called the causal dependency relation and M_0 is a nonempty multiset of conditions, called the initial marking, which satisfy the restrictions:

$$\forall e\varepsilon E,\ \exists\, b\varepsilon B \text{ s.t. } F(b,e) > 0 \text{ and } \forall e\varepsilon E,\ \exists\, b\varepsilon B \text{ s.t. } F(e,b) > 0$$

Thus, we insist that each event "causally depends" on at least one condition and has at least one condition which is causally dependent on it. Winskel [9].

Definition 3: Let Σ be a finite alphabet and K a commutative semi-ring. Then a K-Σ automaton is given by $\mathcal{A} = (Q,\Sigma,F,I,T)$, where Q is a finite set of states, and I and T are K-subsets of Q. I is called the set of initial states, T is called the set of terminal states and F is a K-subset of $Q\times\Sigma\times Q$. If $F(p,\sigma,q) = k \neq 0$ then we say that the edge $p \xrightarrow{k\sigma} q$ is in $\mathcal{A}$ with the label $k\sigma$. Eilenberg [2].

In the discussion below, we assume every state to be terminal. Thus, here a K-Σ automaton is denoted by $\mathcal{A} = (Q,\Sigma,F,I)$.

Definition 4: $(R,\Sigma,\ .,1)$ is a "sum-ordered ring" (So-ring) if Σ is a partial operation on countable families in R, $(R,\bullet,1)$ is a monoid and

i) If $(I_j\colon j\varepsilon J)$ is a partition of the countable set I (with no restriction on how many I_j are empty) then for $f_i:R \to R$ $(i\varepsilon I)$,

$$\Sigma((\Sigma f_i : i\varepsilon I_j) : j\varepsilon J) = \Sigma(f_i : i\varepsilon I)$$

in the sense that if one side is defined then the other necessarily is and then the results are equal.

(ii) The relation $f \leq g$ on $\underline{\underline{C}}(R,R)$ defined to mean $g = f+h$ for some h is antisymmetric, and

(iii) If $f_i : R \to R$ is summable and $g:R \to R$, $h: R \to R$, then hf_ig is summable and

$$\Sigma hf_ig = h(\Sigma f_i)g$$

hold.

A So-ring is an <u>additive domain</u> if also

(iv) Any countable family each of whose finite sub-families is summable is itself summable, and

(v) Given $f_i, g: R \to R$, if Σf_i exists and if each finite subsum is $\leq g$, then $\Sigma f_i \leq g$.

hold. Manes [4].

It is easy to see that $R = \mathbb{N} \cup \{\infty\}$ is an additive domain and any semiring K will become an additive domain if we adjoin a new element ∞.

3. <u>Affinities</u>

In this section, we modify the definition of Petri net by replacing the nonnegative integers $\mathbb{N}$ by an arbitrary commutative semi ring K without proper zero divisors. It is clear that this modification will not affect the results already proved by Winskel, since the theory developed uses only the commutative semiring structure of $\mathbb{N}$.

<u>Definition 5:</u> A Petri net is a 4-tuple $N = (B,E,F,M_o)$, where B is a nonempty set of conditions, E is a set of events, F is a K-subset of

$(B\times E) \cup (E\times B)$. M_0 is a nonempty K-subset of conditions, which satisfy the restrictions:

$$\forall e\in E, \exists\, b\in B \text{ s.t. } F(b,e) \neq 0 \text{ and } \forall e\in E, \exists\, b\in B \text{ s.t. } F(e,b) \neq 0$$

With this definition of Petri net, we are in a position to point out a number of affinities between Petri nets and K-Σ automata. Our first observation in this direction is that given a Petri net, we can construct a K-Σ automaton and vice-versa.

Consider a Petri net $N = (B,E,F,M_0)$. Regarding B as set of states Q, E as set of inputs Σ, M_0 as set of initial states I, define a function F from $(B\times E\times B)$ to K by

$$F(b,e,b') = F(b,e)\, F(e,b')$$

which clearly defines a K-Σ automaton.

On the other hand, consider a K-Σ automaton $\mathcal{A} = (Q,\Sigma,F,I)$. Regarding Q as set of conditions B, Σ as set of events E, I as initial marking M_0, we are constructing a Petri net by defining a function F of Petri net in a following manner:

$$F(p,\sigma,q) = k \Rightarrow F(p,\sigma) = k \text{ and } F(\sigma,q) = k$$

i.e. each input causally depends on at least one state and has at least one state which is causally dependent on it.

Similarly, we can show that the "contact-free" nets (safe nets) of Winskel are essentially the same as "unambiguous" K-Σ automata of Eilenberg. The relevant definitions can be found in Winskel [9] and Eilenberg [2].

4. Spatial Properties

In this section the topological properties of Petri nets will be studied. These are modelled after Shukla and Srivastava [7].

Definition 1: A multifunction from X to Y is a matrix $f: Y \times X \to \mathbb{N}$. Write $f: X \xrightarrow[\mu]{} Y$ to mean f is a multifunction from X to Y.

Definition 2: Let $N = (B,E,F,M_0)$ and $N' = (B',E',F',M_0')$ be nets. A homomorphism from N to N' is a pair of multifunctions (η,β) with $\eta: E \xrightarrow[\mu]{} E'$ and $\beta: B \xrightarrow[\mu]{} B'$ such that $\beta M_0 = M_0'$ and $\forall A \epsilon \mu E$ $^{\bullet}(\eta A) = \beta(^{\bullet}A)$ and $(\eta A)^{\bullet} = \beta(A^{\bullet})$. If η and β are one-one, then N is called a subnet of N'.

Let $N = (B,E,F,M_0)$ be any Petri net and Q be the set of markings of N, then for a subset $R \subseteq Q$, the set of successors of R is defined as $\delta R = \{M' \epsilon Q: M \to M' \text{ for some } M \epsilon R\}$.

A subnet $N' = (B',E',F',M_0')$ is separated (after Bavel [1]) if and only if $\delta(Q-Q') \cap Q' = \phi$, where Q' is the set of markings of N' and N' is connected if and only if it has no separated proper subnets.

For a subset $R \subseteq Q$, the source of R is $\sigma R = \{M \epsilon Q: M \to M'$ for some $M' \epsilon R\}$. It is easy to see that σ is a closure operator for Q. With this consideration, we are able to equip Q with a topology tN for which a subset R of Q is closed if and only if $R = \sigma R$ and consequently a subset $C \subseteq Q$ is open for this topology just when Q-C is closed.

The above topology is saturated in the sense that arbitrary union of closed sets is closed and thus all tN-closed sets are all the open sets for another topology on Q which we denoted by tN*. The closure operator for tN* is δ and tN and tN* is dual in the sense that tN** = tN.

5. Post Script

If we consider the closed interval [0,1] with the operations maximum and minimum in the usual sense, then the semi ring K = [0,1] has been used to model vagueness and nondeterminancy. These K-subsets are precisely Fuzzy sets and they have been very widely used. In this context while working with the general semi ring K, we can think of Petri nets as well as K-Σ automata as models for parallel computing as also models for nondeterministic computing.

Clearly, the unambiguous automata and contact-free Petri nets correspond to the ordinary set theory as the only permitted values are 0 and 1.

We may note that the contact-free Petri nets are introduced because of the difficulties of interpretation (Winskel [8]). Taking [0,1] as semi ring, this may be overcome, since, if F(e,b) = t, then it can be said that b causally depends on e to the degree t. This introduces measures of causal dependency between conditions and events varying from no dependency to full dependency.

The two of course can be regarded as R-sets, where R-sets are precisely defined as K-subsets except that we now employ an additive domain rather than merely a semi ring.

Multisets were introduced to model parallelism, fuzzy sets to model nondeterminism and additive domains to model assertion semantics. The commonality of the mathematical formalism in all the three cases suggests that the three concepts are interlinked in a substantial manner. We are currently investigating these interlinks.

Yet another topology in the context of Petri nets is worthy of mention. Elementary event structures have been defined by Winskel and others (Nielsen, Plotkin and Winskel [6]). Mauri and Brambilla have introduced the concept of left-closed subsets (Definition 2.2 of Mauri and Brambilla [5]). Mauri and Brambilla have observed that the set of all left-closed subsets forms a Heyting algebra (Theorem 3.1 of Mauri and Brambilla [5]).

Heyting algebras are also regarded as formal topological spaces (Johnstone [3]). Taking a clue from here, we can think of collection of left-closed subsets as a topology on the set of events. Then we can easily see that this topology is saturated.

We are investigating the topological and net theoretic properties of this topology.

References

[1] Z.Bavel, The source as a Tool in Automata, Information and Control. 18, 140-155 (1971).

[2] S. Eilenberg, Automata, Languages and Machines, Vol. A. Academic Press, New York (1974).

[3] P.T. Johnstone, Stone Spaces, Cambridge University Press (1982).

[4] E.G. Manes, Weakest preconditions: Categorical insights, Lecture notes in Computer Science, 240, 182-197 (Springer, Berlin, 1985).

[5] G.Mauri and M.Brambilla, On the logic of concurrency and conflict, Second European Workshop on Application and Theory of Petri nets.

[6] M.Nielsen, G.Plotkin and G. Winskel, Petri nets, event structures and domains, Theoretical Computer Science, 13,85-108 (1981).

[7] W.Shukla and A.K. Srivastava, A Topology for Automata: A note, Information and Control, 32, 163-168 (1976).

[8] G.Winskel, Categories of models for concurrency, Lecture notes in Computer Science, 197, 246-267 (Springer, Berlin, 1984).

[9] G.Winskel, Petri nets, Algebras, Morphisms and Compositionality, Information and Computation, 72(3), 197-238 (1987).