LATTICE-VALUED LOGIC AND THREE-VALUED LOGIC

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ABSTRACT

In this paper, we have proved that any two logic formulae are identical in any complemented distributive lattice which contains O, M, and I if and only if they are identical in the three-valued logic, thus it tell us that the simplification of lattice-valued logic formula can be realized by three-valued logic, and the simplification of lattice-valued logic formula becomes much simpler.

KEYWORDS: Lattice-Valued Logic, Three-Valued Logic, Complemented Distributive Lattice, L-Valid, L-Inconsistent.

I. LATTICE-VALUED LOGIC FORMULA

Let the variable set be $\{ x_1, \ldots, x_n \}$.

DEFINITION 1. Let L be a lattice [1], $\leqslant$ a partly ordered relation on L. L is called a complemented lattice if the mapping

$$' : L \longrightarrow L$$

satisfies for all $a, b \in L$ the following conditions:

(1) $(a')' = a$;

(2) If $a \leqslant b$, then $b' \leqslant a'$.

M is called an intermediate element, if $M \in L$ satisfies the following conditions:

(1) $M' = M$ (i.e. M is an immovable point of the mapping $'$);

(2) $\forall a \in L$, $a \leqslant M$ or $M \leqslant a$.

It is easily shown the intermediate element is unique, it is always written as M in this paper.

What follows in the passage, L expresses always a complemented distributive lattice which contains O, M and I. The operation of the supremum and the infimum will be denoted, by "+" and "." respectively (in general, "." is omitted).

DEFINITION 2. The variable x or its complememtary $\bar{x}$ is called literal. Literal product is called phrase, denoted it by P, P' or P_i. Literal sum is called clause, denoted it by C, C' or C_i.

DEFINITION 3. A lattice-valued logic formula denoted $F(x_1, \ldots, x_n)$ is a mapping

$$F : L^n \longrightarrow L$$

We define lattice-valued logic formula generated by $x_1, \ldots, x_n$ recursively as follows:

(a) O, I and x are lattice-valued logic formulae;

(b) if F is a lattice-valued logic formula, then $\bar{F}$ is a lattice-valued logic formula;

(c) if F and G are lattice-valued logic formulae, then F+G and F.G are lattice-valued logic formulae;

(d) the only lattice-valued logic formulae are those given by (a)-(c).

Let F and G be two lattice-valued logic formulae. we define

$$\bar{F}(A) = (F(A))'$$

$$(F.G)(A) = F(A).G(A)$$

$$(F+G)(A) = F(A)+G(A)$$

DEFINITION 4. Let F_1 and F_2 be two lattice-valued logic formulae We say that F_2 contains F_1, denoted by $F_1 \leqslant F_2$, if $\forall A \in L^n$,

$$F_1(A) \leqslant F_2(A)$$

In particular we say that F_1 is L-valid (L-inconsistent) if $\forall A \in L^n$,

$$F_1(A) \geqslant M \ (\leqslant M)$$

II. LATTICE-VALUED LOGIC AND THREE-VALUED LOGIC

LEMMA 1. Let $x \in \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}$; then $x+\bar{x}$ is L-valid and $x.\bar{x}$ L-inconsistent.

LEMMA 2. (a) A clause C is L-valid if and only if it contains a pair of variables $(x_i, \bar{x}_i)$;

(b) A phrase P is L-inconsistent if and only if it contains a pair of variables $(x_i, \bar{x}_i)$.

Setting

$$L^* = \{ O, M, I \}.$$

It is clear that $(L^*, \leqslant)$ is a sublattice of L. We say that a lattice-valued logic formula is three-valued logic formula if we use L^* instead of L in definition 3.

LEMMA 3. Let P be a phrase, C a clause; then $P \leqslant C$ (in lattice-valued logic) if and only if $P \leqslant C$ (in three-valued logic).

PROOF. The necessity is clear, we need establish only sufficiency. We divide it into two cases.

Case 1. There is the same literal in P and C

Let literal x be contained in P and C; then

$$P \leqslant x \leqslant C$$

Case 2. There is no the same literal in P and C

(1) If C contains a pair of variables $(x_i, \bar{x}_i)$ and so does P, then

$$P \leqslant M \leqslant C$$

by lemma 2.

(2) If C contains a pair of variables $(x_i, \bar{x}_i)$, and P does not contain any pair of variables $(x, \bar{x})$. Setting

$$A = (a_1, \ldots, a_n) \in L^{*n}$$

and it satisfies the following condition:

$$a_j = \begin{cases} O, & \bar{x}_j \text{ is contained in P;} \\ I, & x_j \text{ is contained in P;} \\ M, & \text{otherwise,} \end{cases}$$

then $P(A) = I, C(A) = M$ in contradiction with $P \leqslant C$ (in three-valued logic formula). Therefore, (2) does not hold.

(3) If C does not contain any pair of variables $(x, \bar{x})$, and P contains a pair of variables $(x_i, \bar{x}_i)$. Setting

$B = (b_1, \ldots, b_n) \in L^{*n}$

and it satisfies the following condition

$$b_j = \begin{cases} O, & x_j \text{ is contained in } C; \\ I, & \bar{x}_j \text{ is contained in } C; \\ M, & \text{otherwise,} \end{cases}$$

then $P(B) = M, C(B) = O$ in contradiction with $P \leqslant C$ (in three-valued logic).Therefore,(3) does not hold.

(4) If C and P do not contain any pair of variables $(x,\bar{x})$. Setting

$D = (d_1, \ldots, d_n) \in L^{*n}$

and satisfies the condition

$$d_i = \begin{cases} O, & x_i \text{ is contained in } C \text{ or } \bar{x}_i \text{ in } P; \\ I, & \bar{x}_i \text{ is contained in } C \text{ or } x_i \text{ in } P; \\ M, & \text{otherwise,} \end{cases}$$

then $C(D) = O, P(D) = I$ in contradiction with $P \leqslant C$ (in three-valued logic).Therefore,(4) does not hold.

To sum up,the case 1 and the case 2 (1) hold only. So the sufficiency is proved. Q.E.D.

By the proof of lemma 3,we have

COROLLARY 1.Let P be a phrase,and C a clause; then $P \leqslant C$ if and only if there is the same literal in P and C, or P and C contain complementary pair.

THEOREM.Let F and G be two lattice-valued logic formulae.Then $F \leqslant G$ (in lattice-valued logic) if and only if $F \leqslant G$ (in three-valued logic).

PROOF.The necessity is clear, we need prove only sufficiency. First all we note the fact that each lattice-valued logic formula can be written either in the disjunctive normal form or the conjunctive normal form.This is obvious,due to the properties of L. Therefore,we can write

$F = P_1 + \ldots + P_m$

$C = C_1 \ldots C_t$

Where,P_i is a phrase ($i = 1,2,\ldots,m$),C_j is a clause ($j = 1,2,\ldots, t$).To prove

$P_1 + \ldots + P_m \leqslant C_1 \ldots C_t$ (in lattice-valued logic)

By

$P_1 + \ldots + P_m \leqslant C_1 \ldots C_t$ (in three-valued logic)

We have

$\forall i \in \{1,\ldots,m\}, \forall j \in \{1,\ldots,t\}, P_i \leqslant C_j$ (in three-valued logic).

By lemma 3,we have

$\forall i \in \{1,\ldots,m\}, \forall j \in \{1,\ldots,t\}, P_i \leqslant C_j$ (in lattice-valued logic),so

$\forall j \in \{1,\ldots,t\}, P_1 + \ldots + P_m \leqslant C_j$ (in lattice-valued logic).

Therefore,$P_1 + \ldots + P_m \leqslant C_1 \ldots C_t$ (in lattice-valued logic).

Q.E.D.

COROLLARY 2.Let F and G be two lattice-valued logic formulae, then,$F = G$ (in lattice-valued logic) if and only if $F = G$ (in three-valued logic).

By the order relation "$\leqslant$" of real number,and we define that $\forall a \in [0,1], (a)' = 1-a$,then $([0,1], \leqslant, ')$ is a complemented distributive lattice.

By corollary 2 we have

COROLLARY 3. Let F and G be two fuzzy logic formulae [2-5], then F = G (in fuzzy logic) if and only if F = G (in three-valued logic — (0,0.5,1)).

III. CONCLUSION

By the conclusion in this paper, we want to study the equality property of logic formula in lattice-valued logic (in special, in fuzzy logic), if and only if we study it only in three-valued logic, thus the infinite value problem becomes the three-valued problem, and the simplification, decomposition, composition and combinational switching systems of lattice-valued logic formula (in special, fuzzy logic formula) become much simpler.

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