SOME REMARKS ON FUZZY IMPLICATION OPERATORS*

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1. Introduction

There exist several constructions for fuzzy implication operators via conjunctions. In this paper we present a unifying approach to the generation of implications and we prove that for a rather general class of conjunctions (will be called f-norms) the generation process is closed. Besides, all well-known families of fuzzy implications are within our framework.

2. Background

Let $I = [0,1]$ and $I_0 = (0,1)$. A function $T: I \times I \to I$ is said to be a *t-norm* iff T is commutative, associative, non-decreasing and $T(a,1) = a$, $\forall\, a \in I$. A t-norm T is *Archimedean* iff it is continuous and $T(a,a) < a$, $\forall\, a \in I_0$.

A function $n: I \to I$ will be called *negation* if n is non-increasing and $n(0) = 1$, $n(1) = 0$. A negation is called *strict* if n is continuous and decreasing. A strict negation is called *strong* if $n(n(a)) = a \quad \forall\, a \in I$.

Let $\circ$ be any binary operation on I. Let us define four operation on I in the following way (see [1]):

$a\ [\mathcal{A}(\circ)]\ b := b \circ a$

$a\ [\mathcal{S}_n(\circ)]\ b := n(a \circ n(b))$, where n is a negation.

$a\ [\mathcal{I}(\circ)]\ b := \sup \{ s \in I \ ;\ a \circ s \leqq b \}$

$:= 0$ if there is no $s \in I$ such that $a \circ s \leqq b$.

$a\ [\mathcal{V}_n(\circ)]\ b := n(b) \circ n(a)$.

*This work has been partially supported by OTKA-27-5-606.

Dubois and Prade proved the following theorem in [1] :

If $*$ is the t-norm 'min' or $*$ is an Archimedean t-norm then we have

$$\mathcal{I} \circ \mathcal{S}_n \circ \mathcal{I}\,(*) \;=\; \mathcal{S}_n\,(*) \qquad (1)$$

and

$$\mathcal{I} \circ \mathcal{A} \circ \mathcal{S}_n \circ \mathcal{I}\,(*) \;=\; \mathcal{D}_n \circ \mathcal{I}\,(*)\,, \qquad (2)$$

where n is a strong negation .

They noticed that this result is not "a complete answer to the problem of generating multivalued implication functions". For example, the following class defined by $a \to b = S(n(a),T(a,b))$ (where T is a t-norm, S is a t-conorm and n is a negation) or the implication $a \to b = b^a$ proposed by Yager [5] are outside of the above-mentioned framework.

3. F-norms and related concepts

We recall that a function $w: I \times I \to I$ is called *weak t-norm* if $w(a,1) \leqq a$, $w(1,b) = b$ $\forall$ $a,b \in I$ and $w(a,b) \leqq w(c,d)$ when $a \leqq c$, $b \leqq d$ (see [2] for details and applications to strict preference relations). In this section we generalize this notion further.

__Theorem 1.__ Let $*$ be a given binary operation on I.

(a) If (1) holds then $*$ has the following properties:

$1*0 = 0*1 = 0*0 = 0$; $1*1 = 1$. (3)

$1*b > 0$ when $b > 0$. (4)

$a*b \leqq a*d$ when $b \leqq d$. (5)

(b) If (2) holds then $*$ has the properties (3) - (4) and

$a*b \geqq c*b$ when $a \leqq c$. □ (6)

The conditions (3) - (5) are necessary for a binary operation to fulfill the relation (1) and similarly, (3)-(6)

are necessary for validity of (2) . However, it is easy to see that these conditions are not sufficient. Indeed, let $a*b = T_W(a,b)$ the well-known weakest t-norm. In this case (1) is false. Hence, it seems to be necessary to require some type of continuity in the second argument of $*$. On the other hand, it is also reasonable to suppose that $\{s; a*s \leqq b\}$ is non-empty $\forall\, a,b \in I$, i.e., $a*0 = 0$ for every $a \in I$.

Following these simple ideas, we introduce an operation in I playing a central role in this paper.

__Definition 1.__ (a) A binary operation $*$ on the closed unit interval I is called ***f-norm*** if it satisfies the following conditions:

(i) $0*1 = 0$, $a*0 = 0$ for every $a \in I$.

(ii) $1*b > 0$ for every $b > 0$; $1*1 = 1$.

(iii) $a*b \leqq a*d$ when $b \leqq d$.

(b) If $*$ is an f-norm then $a[\mathcal{I}(*)]b$ is called the *right pseudocomplement* of $*$.

(For t-norms and weak t-norms see [4] and [2], respectively.) We note that (i) and (iii) imply $0*b = 0$ for every $b \in I$. Moreover, it is clear that every t-norm as well as every weak t-norm is an f-norm.

Denote $\mathcal{F}$ the class of all f-norms satisfying the following (technical) requirement:

$$a*x \text{ is left-continuous with respect to } x \text{ on } I \text{ for every } a \in I. \qquad (7)$$

Let now $\sim : I\times I \to I$ such that

(a) $a\sim x$ is right-continuous with respect to x on I for every $a \in I$.

(b) $1\sim b < 1$ when $b < 1$; $a\sim 1 = 1$ for every $a \in I$.

(c) $0\sim 0 = 1$, $1\sim 1 = 1$, $1\sim 0 = 0$.

(d) If $b \leqq d$ then $a\sim b \leqq a\sim d$.

Denote $\mathcal{H}$ the set of functions $\sim$ which fulfill conditions (a) - (d). It is easy to see that $* \in \mathcal{F}$ implies $\mathcal{I}(*) \in \mathcal{H}$.

For the sake of simplicity, denote $\mathcal{U} = \mathcal{S}_n \circ \mathcal{I} \circ \mathcal{S}_n$. Now we can state the following result extending Theorem 5 of [3] and Proposition 2.3 of [2].

Theorem 2. (a) If $* \in \mathcal{F}$ then $* = \mathcal{U} \circ \mathcal{I}(*)$.
(b) If $\sim \in \mathcal{H}$ then $\sim = \mathcal{I} \circ \mathcal{U}(\sim)$.□

Examples for f-norms:

(a) t-norms, weak t-norms and the following types of means (among others) are f-norms:

$$M_\lambda^r(a,b) = \begin{cases} [\lambda a^r + (1-\lambda)b^r]^{1/r} & \text{if } ab > 0 \\ 0 & \text{if } ab = 0 \end{cases}, \text{ where } r \in \mathbb{R}\backslash\{0\}.$$

It is easy to see (by taking limits) that

$$M_\lambda^{-\infty}(a,b) = \min(a,b), \quad M_\lambda^0(a,b) = \begin{cases} a^\lambda b^{1-\lambda} & \text{if } ab > 0 \\ 0 & \text{if } ab = 0 \end{cases} \text{ and}$$

$$M_\lambda^{+\infty}(a,b) = \begin{cases} \max(a,b) & \text{if } ab > 0 \\ 0 & \text{if } ab = 0 \end{cases}.$$

(b) Let $S(a,b)$ be any t-conorm . Then the operation $*_S$ defined by $a *_S b = \begin{cases} S(a,b) & \text{if } ab > 0 \\ 0 & \text{if } ab = 0 \end{cases}$ is also an f-norm.

4. Closure theorems

In this section we give generalized versions of the Theorem of [1] as well as Theorem 4.3 and 4.4 of [2].

Theorem 3. If $* \in \mathcal{F}$ then the relation (1) is true.

Proof. By definition we have the following chain of equality : $a[\mathcal{I}(\mathcal{S}_n(\mathcal{I}(*)))]b = n[\inf\{y;\ a[\mathcal{I}(*)]y \geqq n(b)\}] = a[\mathcal{S}_n(*)]b$, by Theorem 2. □

<u>**Theorem 4.**</u> If $* \in \mathcal{I}$ is commutative then the relation (2) holds .

Proof. $a[\mathcal{I}(\mathcal{A}(\mathcal{S}_n(\mathcal{I}(*))))]b = \sup \{ x \ ; \ x[\mathcal{I}(*)]n(a) \geqq n(b) \} =$ $= \sup \{ x \ ; \ n(b)*x \leqq n(a) \} = a[\vartheta_n(\mathcal{I}(*))]b$. Here we used also Theorem 2. □

It is obvious that our approach contains implications mentioned at the end of the section 2 as well as the three types of Weber [4].

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