THE ENTROPY OF FUZZY DYNAMICAL SYSTEMS

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In this contribution the entropy of an F-state and the entropy of fuzzy dynamical systems are studied. The main properties of such quantities are stated. The connection with the classical cases is also mentioned.

1. SOME DEFINITIONS AND NOTATIONS

An F-quantum space is a couple (X,M), where $X \neq \emptyset$ and $M \subset \langle 0,1\rangle^X$ satisfies the following conditions: (1.1) if $1(x) = 1$ for any $x \in X$, then $1 \in M$, (1.2) if $f \in M$, then $f' := 1 - f \in M$; (1.3) if $1/2\,(x) = 1/2$ for any $x \in X$, then $1/2 \notin M$; (1.4) $\bigvee_{n=1}^{\infty} f_n := \sup_n f_n \in M$ for any $\{f_n\}_{n=1}^{\infty} \subset M$.

This structure has been suggested by Riečan ([1], [2]) as an alternative model for quantum mechanics. In the set M we define the relation $\leq$ in the following way: $f \leq g$ iff $f(x) \leq g(x)$ for each $x \in X$. In accordance with the theory of quantum logics we say that $f, g \in M$ are orthogonal (we write $f \perp g$), if $f \leq 1 - g$.

An F-state on an F-quantum space (X,M) is a mapping $m : M \longrightarrow \langle 0,1\rangle$ satisfying the following conditions: (1.5) $m(f \vee (1 - f)) = 1$ for every $f \in M$; (1.6) if $\{f_n\}_{n=1}^{\infty}$ is a sequence of pairwise orthogonal fuzzy subsets from M, then $m(\bigvee_{n=1}^{\infty} f_n) = \sum_{n=1}^{\infty} m(f_n)$.

Example 1.1. Let $(X, \mathcal{S}, P)$ be a probability space. Put $M = \{I_A ; A \in \mathcal{S}\}$ (I_A is the indicator of the set $A \in \mathcal{S}$). Then (X, M) is an F-quantum space and the mapping $m : M \to \langle 0,1 \rangle$ defined by $m(I_A) = P(A)$ is an F-state on (X, M).

2. THE ENTROPY OF AN F-STATE

A finite set $\mathcal{A} = \{f_1, \dots, f_n\}$, $f_i \in M$, we shall call an orthogonal m-partition of the unit, if for each $f_i, f_j \in \mathcal{A}$, $i \neq j$, it holds $f_i \perp f_j$ and $m(\bigvee_{i=1}^{n} f_i) = 1$. In the set $\mathcal{P}$ of all orthogonal m-partitions of the unit one can define the operation $\vee$ in the following way: if $\mathcal{A}, \mathcal{B} \in \mathcal{P}$, then $\mathcal{A} \vee \mathcal{B} := \{f \wedge g ; f \in \mathcal{A}, g \in \mathcal{B}\}$. Further we define in $\mathcal{P}$ the partial ordering „ $\leq$ " : $\mathcal{A} \leq \mathcal{B}$ if there exists $\mathcal{C} \in \mathcal{P}$ such that $\mathcal{B} = \mathcal{A} \vee \mathcal{C}$. Each $\mathcal{A} \in \mathcal{P}$ in the sense of the classical probability theory represents the random experiment with finite number of outcomes with the probability distribution $p_i = m(f_i)$, $f_i \in \mathcal{A}$, since $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} m(f_i) = m(\bigvee_{i=1}^{n} f_i) = 1$.

Definition 2.1. The entropy $H_m(\mathcal{A})$ of an orthogonal m-partition $\mathcal{A} = \{f_1, \dots, f_n\}$ in the F-state m we define by Shannon's formula:

(2.1) $H_m(\mathcal{A}) = - \sum_{i=1}^{n} F(m(f_i))$, where $F : \langle 0, \infty) \to R$,

$$F(x) = \begin{cases} x \log x, & \text{if } x > 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Theorem 2.1. The entropy $H_m : \mathcal{P} \to R$ has the following properties: (2.2) $H_m(\mathcal{A}) \geq 0$ for every $\mathcal{A} \in \mathcal{P}$;

(2.3) if $\mathcal{A}, \mathcal{B} \in \mathcal{P}$, $\mathcal{A} \leq \mathcal{B}$, then $H_m(\mathcal{A}) \leq H_m(\mathcal{B})$;

(2.4) $H_m(\mathcal{A} \vee \mathcal{B}) \leq H_m(\mathcal{A}) + H_m(\mathcal{B})$ for every $\mathcal{A}, \mathcal{B} \in \mathcal{P}$.

Example 2.1. Let $X = \langle 0,1\rangle$, $f : X \to X$, $f(x) = x$ for every $x \in X$, $M = \{f, f', f \vee f', f \wedge f', 0, 1\}$, $m(1) = m(f \vee f') = 1$, $m(0) = m(f \wedge f') = 0$, $m(f) = m(f') = 1/2$. Then the set $\mathcal{A} = \{f, f'\}$ is only orthogonal m-partition of the unit with the non-zero entropy and hence $h(m) = H_m(\mathcal{A}) = \log 2$.

Example 2.2. Let $(X, \mathcal{S}, P)$ be a finite probability space. If we define (X,M) and m as in Example 1.1, then the entropy of an m-partition $\mathcal{A} = \{I_{A_1}, \dots, I_{A_k}\}$ is the number $H_m(\mathcal{A}) = -\sum_{i=1}^{k} F(P(A_i))$ and the entropy of the F-state m is $h(m) = -\sum_{i=1}^{n} F(p_i)$, what is the Shannon entropy of the probability distribution $\overline{p} = \{p_1, \dots, p_n\}$ on X.

3. THE ENTROPY OF FUZZY DYNAMICAL SYSTEMS

By a fuzzy dynamical system we shall mean the quadruple $(X,M,m,\mathcal{U})$ where (X,M) is an F-quantum space, m is an F-state on (X,M) and $\mathcal{U} : M \to M$ is a σ-homomorphism fulfilling the condition:

(3.1) $m(\mathcal{U} f) = m(f)$ for every $f \in M$.

Example 3.1. Let $(X, \mathcal{S}, P, T)$ be a dynamical system in the sense of the classical probability theory. If we define (X,M) and m as in Example 1.1 and the mapping $\mathcal{U} : M \to M$ by $\mathcal{U}(I_A) = I_{T^{-1}(A)}$, then $(X,M,m,\mathcal{U})$ is a fuzzy dynamical system. In this case we shall say, that $(X,M,m,\mathcal{U})$ is induced by $(X, \mathcal{S}, P, T)$.

We define $\mathcal{U}^2 = \mathcal{U} \circ \mathcal{U}$ and by mathematical induction $\mathcal{U}^n = \mathcal{U} \circ \mathcal{U}^{n-1}$, $n = 1,2,\dots$, where $\mathcal{U}^0$ is the identical mapping on M. For every $\mathcal{A} \in \mathcal{P}$ $\mathcal{U}^n \mathcal{A} := \{\mathcal{U}^n(f);\ f \in \mathcal{A}\} \in \mathcal{P}$ and it holds $H_m(\mathcal{U}^n \mathcal{A}) = H_m(\mathcal{A})$.

Definition 3.1. Let $(X,M,m,\mathcal{U})$ be a fuzzy dynamical system. Then for every $\mathcal{A}\in\mathcal{P}$ we define $h_m(\mathcal{U},\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H_m(\bigvee_{j=0}^{n-1} \mathcal{U}^j\mathcal{A})$. The entropy of fuzzy dynamical system $(X,M,m,\mathcal{U})$ we define by $h_m(\mathcal{U}) = \sup\{h_m(\mathcal{U},\mathcal{A});\ \mathcal{A}\in\mathcal{P}\}$.

Theorem 3.1. Let $(X,\mathcal{S},P,T)$ be a dynamical system in the classical sense and $(X,M,m,\mathcal{U})$ be a fuzzy dynamical system induced by $(X,\mathcal{S},P,T)$. Then $h_m(\mathcal{U}) = h(T)$, where $h(T)$ is the Kolmogorov - Sinaj entropy ([3]) of the dynamical system $(X,\mathcal{S},P,T)$.

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