ON THE KURZWEIL INTEGRAL OF FUZZY FUNCTIONS AND REAL FUNCTIONS

Anna Paruleková
TECHNICAL UNIVERSITY
Vrbická 1944,031 01 Liptovský Mikuláš
CZECHOSLOVAKIA

The paper deals with generalizing of Kurzweil integral for fuzzy mapping,i.e.mapping with values in the set of fuzzy numbers and gives a relationship between such a fuzzy integral and real integral.Here we only introduce briefly the definition of integral in the Kurzweil sense.

Let $f: [a,b] \to R$ be a real function and $D = \{(E_i, x_i), i=1,\dots,n\}$ be such a partition of $[a,b]$ that $x_i \in E_i$, E_i $(i=1,\dots,n)$ are compact subintervals of $[a,b]$ such that $\mathrm{Int}\, E_i \cap \mathrm{Int}\, E_j = \emptyset$ for $i \neq j$ $(i,j=1,\dots,n)$ and $\bigcup_{i=1}^{n} E_i = [a,b]$. The integral sum of f for the partition D have the form

$$S(f,D) = \sum_{i=1}^{n} f(x_i)\lambda(E_i) \qquad (1)$$

where λ is the Lebesgue measure.

Let $\Delta : [a,b] \to (0,\infty)$ be a function. A convenient partition of $[a,b]$ with respect to Δ is such a partition D that

$$E_i \subset (x_i - \Delta(x_i), x_i + \Delta(x_i)), i=1,\dots n \qquad (2)$$

The set of all convenient partitions of $[a,b]$ with respect to Δ will be denoted by $\mathcal{D}(\Delta)$. It is easy to prove that for any fonction $\Delta : [a,b] \to (0,\infty)$ is $\mathcal{D}(\Delta) \neq \emptyset$

Futher we will consider a fuzzy mapping $f: [a,b] \to L(R)$, where L(R) denote the set of fuzzy numbers,i.e.the set of functions $\mu : R \to [0,1]$ that satisfy the following conditions:

1) There is $x_0 \in R$ such that $\mu(x_0) = 1$
2) The set $\mu_\alpha = \{x \in R: \mu(x) \geq \alpha\}$ is convex for all $\alpha \in (0,1]$
3) μ is upper semicontinuous function
4) There is a compact set $K \subset R$ such that $\{x \in R: \mu(x) > 0\} \subset K$

We can define on the set L(R) a metric d by the following

formula: $d(\mu,\nu)=\sup_{\alpha\in[0,1]}\left\{\max\left[|c_\alpha - a_\alpha| , |d_\alpha - b_\alpha|\right]\right\}$,
where $\mu,\nu\in L(R)$ and $\mu_\alpha=[a_\alpha, b_\alpha]$, $\nu_\alpha=[c_\alpha, d_\alpha]$.
Puri and Ralescu proved that $(L(R),d)$ is a complete metric space.

D e f i n i t i o n 1. A fuzzy mapping $f: [a,b]\to L(R)$ is integrable (in the Kurzweil sense), if
$\exists c\in L(R)\ \forall\ \varepsilon>0\ \exists\ \Delta: [a,b]\to(0,\infty): \forall D\in\mathcal{D}(\Delta)$:
$d(S(f,D),c)<\varepsilon$. The fuzzy number c is called the Kurzweil fuzzy integral of f and it is denoted by $\int_a^b f d\lambda$.

We will define for every $\alpha\in(0,1]$ functions $P_\alpha, Q_\alpha: L(R)\to R$ by the formulas

$$P_\alpha\mu = \min\{x:\mu(x)\geq\alpha\},\quad Q_\alpha\mu = \max\{x:\mu(x)\geq\alpha\},$$

$\mu\in L(R)$, i.e. $P_\alpha f, Q_\alpha f : [a,b]\to R$.

P r o p o s i t i o n 1. Let $f: [a,b]\to L(R)$ be a fuzzy function, $P_\alpha f, Q_\alpha f : [a,b]\to R$. Then f is integrable (in the Kurzweil sense), if only if $P_\alpha f, Q_\alpha f$ are integrable.

R e m a r k 1. The integrals of functions $P_\alpha f, Q_\alpha f$ ($\alpha\in(0,1]$) are the Perron integrals.

P r o p o s i t i o n 2. A fuzzy function $f: [a,b]\to L(R)$ is Kurzweil integrable if only if there exists the Perron integral of $Q_1 f$ and there exist the Lebesgue integrals of $Q_\alpha f - Q_1 f$, $Q_1 f - P_\alpha f$.

T h e o r e m 1. Let $(f_n)_n$ be a sequence of integrable fuzzy mappings on $[a,b]$, $f_n\leq f_{n+1}$ (n=1,2,,,,),
$f(x) = \lim_{n\to\infty} f_n(x)$ for every $x\in[a,b]$ and let the sequence of fuzzy numbers $\left(\int_a^b f_n d\lambda\right)_n$ be convergent with respect to the metric. Then f is integrable fuzzy mapping, too, and

$$\int_a^b f d\lambda = \lim_{n\to\infty}\int_a^b f_n d\lambda \text{ holds.}$$

R e m a r k 2. $f_n\leq f_{n+1}$ means that for all $x\in[a,b]$ is $f_n(x)\leq f_{n+1}(x)$ with respect to relation "$\leq$" on L(R) ([4]).

R e m a r k 3. $\lim_{n\to\infty} f_n(x) = f(x)$ means that for all $x\in[a,b]$:
$\forall\ \varepsilon>0\ \exists N=N(\varepsilon,x): \forall\ n>N: d(f_n(x),f(x))<\varepsilon$ ([4]).

T h e o r e m 2. Let $(f_n)_n$ be a sequence of integrable fuzzy mappings on $[a,b]$, $\lim_{n\to\infty} f_n(x) = f(x), x \in [a,b]$. Let there exist the integrable functions v,w such that $v \leq f_n \leq w$ $(n=1,2,\ldots)$ for every $x \in [a,b]$. Then f is integrable function, too, and $\lim_{n\to\infty} \int_a^b f_n d\lambda = \int_a^b f d\lambda$ holds. (v,w,f are fuzzy f.)

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