ON THE REPRESENTATION OF OBSERVABLES IN FUZZY - QUANTUM SPACES

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Let P be a quantum logic, $x,y : \mathcal{B}(R) \to P$ be two observables and $x(\mathcal{B}(R)) \subset y(\mathcal{B}(R))$, then there is a Borel measurable function $T : R \to R$ such that $x = y \circ T^{-1}$ (see e.q. [1], [2]). In this note, we present a generalization of this lemma for observables on quantum logic to σ- homomorphisms on weakly orthocomplemented σ- posets. As a special case we obtain also a representation lemma in F - quantum spaces ([3], [4]). This result enables us to prove a variant of the ergodic theorem in F - quantum spaces ([5], [6]) and probably some other limit theorems, too ([7]).

We shall say that a partially ordered set P with a mapping $a \to a'$ is a weakly orthocomplemented σ- poset, if (i) $(a')' \geq a$ for every $a \in P$; (ii) if $a,b \in P$, $a \leq b$, then $b' \leq a'$; (iii) if $(a_i)_i \subset P$, $a_i \leq a'_j$ $(i \neq j)$, then there exists $\bigvee_i a_i$ in P; (iv) $a \neq a'$ for every $a \in P$. These posets were studied, e.q. in [8].

A set F of functions $f : X \to \langle 0,1 \rangle$ is an F - quantum space, if the following conditions are satisfied : a) F contains the constant function 0 and does not contain the constant function 1/2; b) if $f \in F$, then $f' = 1 - f \in F$; c) if $f_n \in F$ $(n = 1,2,\dots)$, then $\sup_n f_n \in F$.
It is clear that every F - quantum space satisfies the above assumptions (i) - (iv).

J. Pykacz ([9]) suggested to substitute the property c) in

F - quantum space, by a weaker one : c_1) if $f_n \in F$ $(n = 1,2,\dots)$ and $f_n \leq f_m' = 1 - f_m$ $(n \neq m)$, then $\sup_n f_n \in F$.
Evidently, also the weaker form of an F - quantum space satisfies the above assumptions. It is simple to show that it is not true that $f \vee f' = 1$, in general.

DEFINITION 1. Let $\mathcal{B}$ denote a σ- algebra of subsets of a nonvoid set Y. Let P be a weakly orthocomplemented σ- poset. A mapping $x : \mathcal{B} \to P$ is called σ- homomorphism if

1) $x(E^c) = (x(E))'$ for every $E \in \mathcal{B}$;

2) $x(E) \leq (x(F))'$ if $E, F \in \mathcal{B}$, $E \cap F = \emptyset$;

3) if $E_n \in \mathcal{B}$ $(n = 1,2,\dots)$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, then $x(\bigcup_n E_n) = \bigvee_n x(E_n)$.

In particular, if $\mathcal{B} = \mathcal{B}(R)$ ($\mathcal{B}(R)$ is the set of all Borel subsets in R), then σ- homomorphism x is called an observable.
In [10] the following theorem is to prove.

THEOREM 1. Let $\mathcal{B}$ be a σ- algebra of subsets of a set $Y \neq \emptyset$ containing a countable generator of $\mathcal{B}$. Let P be a weakly orthocomplemented σ- poset. Let $y, z : \mathcal{B} \to P$ be σ- homomorphisms such that $y(E) = y(\emptyset)$ iff $E = \emptyset$, and $z(\mathcal{B}) \subseteq y(\mathcal{B})$. Then there is a $\mathcal{B}$-measurable mapping $T : Y \to Y$ such that $z = y \circ T^{-1}$.

In order to prove the more general assertion than Theorem 1, we need the further notions.

DEFINITION 2. Let $\mathcal{B}$ be a σ- algebra of subsets of a set Y. A set $\mathcal{A} \subseteq \mathcal{B}$ is said to be a maximal σ- filter, if (i) $\mathcal{A} \neq \emptyset$; (ii) $G_n \in \mathcal{A}$, $n \geq 1$, implies $\bigcap_n G_n \in \mathcal{A}$; (iii) if $G \subset H$, $G \in \mathcal{A}$, $H \in \mathcal{B}$, then $H \in \mathcal{A}$; (iv) $\mathcal{A}$ contains exactly one of the elements A, A^c for every $A \in \mathcal{B}$.

DEFINITION 3. A σ- algebra $\mathcal{B}$ is said to be σ- perfect, if any maximal σ- filter $\mathcal{A}$ of $\mathcal{B}$ is determined by some point $t \in Y$, i. e. $\mathcal{A} = \{E \in \mathcal{B} : t \in E\}$, for some $t \in Y$.

THEOREM 2. Let $\mathcal{B}$ be a σ- algebra of subsets of a set $Y \neq \emptyset$. Let P be a weakly orthocomplemented σ- poset. Let $y, z : \mathcal{B} \to P$ be arbitrary two σ- homomorphisms, such that $y(E) = y(\emptyset)$ iff $E = \emptyset$, and $z(\mathcal{B}) \subseteq y(\mathcal{B})$. Then there is a $\mathcal{B}$ - measurable mapping $T : Y \to Y$, such that $z = y \circ T^{-1}$ iff $\mathcal{B}$ is σ- perfect.

For the proof of the Theorem 2 see [11].

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