INDEFINITE INTEGRAL IN FUZZY QUANTUM SPACES

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A new model for quantum mechanics was suggested by A. Dvurečenskij and the author in [1] and [2]. This model was further developped e.g. in [3] - [5]. In [6] - [8] a calculus for observables was constructed. In a framework of this calculus we shall construct the indefinite integral of an observable and prove its σ-additivity. Another approach to the problem is given in [10].

1. Notions and notation. There are three basic notions in the F-quantum spaces theory: F-quantum space, F-observable and F-state.

F-quantum space is a family $F \subset \langle 0, 1\rangle^X$ of real functions satisfying the following properties: 1. If $f \in F$, then $f' = 1 - f \in F$. 2. If $f_n \in F$ $(n=1,2,\dots)$, then $\bigvee_n f_n = \sup_n f_n \in F$.

F-observable is a σ-homomorphism from the σ-algebra of Borel subsets of R to F, i.e. a mapping satisfying the following two properties: 1. $x(E') = x(E)'$ for every $E \in B$. 2. $x(\bigcup_n E_n) = \bigvee_n x(E_n)$ for every $E_n \in B$ $(n=1,2,\dots)$.

F-state is a mapping $m: F \to \langle 0, 1\rangle$ defined on an F-quantum space F and satisfying the following two conditions: 1. $m(a \vee a') = 1$ for every $a \in F$. 2. If $a_n \in F$ $(n=1,2,\dots)$ and $a_i \leq a_j'$ $(i \neq j)$, then $m(\bigvee_n a_n) = \sum_n m(a_n)$.

Recall that the last definition due to Piasecki [9] inspired our investigations.

2. Indefinite integral. A classical analogy of an observable is a random variable ξ defined on a probability space (Ω, S, P). To every random variable ξ and F-observable x can be assigned by

the formula $x(E) = \xi^{-1}(E)$. If x is an F-observable and m is an F-state, then the composite mapping $m \circ x$ is a probability measure. We shall denote it by m_x, hence $m_x(E) = m(x(E))$, $E \in B$. An observable x is integrable if there exists the mean value $m(x) = \int_R t\, dm_x(t)$, what is also in a full agreement with the classical case. Our aim is to define the indefinite integral $\int_a x\, dm$. We shall follow also the classical case, where $\int_A dP = \int \chi_A \xi\, dP$. Therefore we must investigate the preimages $(\xi \chi_A)^{-1}(E)$, $E \in B$. This investigation leads to the following definition.

If $x: B \longrightarrow F$ is an F-observable, then for every $a \in F$ and every Borel set $E \in B$ we define

$$x_a(E) = \begin{cases} a \wedge (x(E) \vee a'), & \text{if } 0 \notin E \\ a' \vee (x(E) \wedge a), & \text{if } 0 \in E \end{cases}$$

Lemma. The mapping $x_a: B \longrightarrow F$ is an F-observable for any $a \in F$.

Theorem. If x is an integrable F-observable, then the function $\mu: F \longrightarrow R$ defined by $\mu(a) = m(x_a) = \int_R t\, dm_{x_a}(t)$ is σ-additive, i.e. $a_i \in F(i=1,2,\ldots)$, $a_i \leq a_j'$ $(i \neq j)$ implies $\mu(\bigvee_i a_i) = \sum_i \mu(a_i)$.

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