ERGODIC THEORY ON FUZZY QUANTUM SPACES

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For fuzzy quantum spaces we define the basic notions of ergodic theory and we generalize the Birkhoff individual ergodic theorem, maximal ergodic theorem and the Poincaré recurrence theorems.

There exists (Riečan, B. [1]) an axiomatic model of quantum mechanics which is based on the ideas of the fuzzy set theory:

DEFINITION 1. <u>A fuzzy quantum space</u> is a couple (X,M), where X is a nonempty set and $M \subset [0,1]^X$ such that the following conditions are satisfied:

(i) if $[1]_X(x) = 1$ for any $x \in X$, then $[1]_X \in M$;

(ii) if $a \in M$, then $a^{\perp} := 1 - a \in M$;

(iii) if $[1/2]_X(x) = 1/2$ for any $x \in X$, then $[1/2]_X \notin M$;

(iv) $\bigcup_{n=1}^{\infty} a_n := \sup_n a_n \in M$, for any $\{a_n\}_{n=1}^{\infty} \subset M$;

By $\bigcap_n a_n$ we mean $\inf a_n$.

The system M is called in the fuzzy sets theory a soft σ-algebra (Piasecki, K. [2]).

DEFINITION 2. <u>An F-state</u> of a fuzzy quantum space (X,M)

is a mapping $m: M \to [0,1]$ such that

(i) $m(a \cup (1 - a)) = 1$ for every $a \in M$;

(ii) if $a_i \in M$ $(i = 1, 2, \dots)$ and $a_i \leq 1 - a_j$ $(i \neq j)$ then

$$m(\bigcup_i a_i) = \sum_i m(a_i).$$

In the fuzzy set theory the mapping m is called a P-measure and M is a soft σ-algebra (Piasecki, K. [2]).

DEFINITION 4. An F-observable on a fuzzy quantum space (X,M) is a mapping $x: B(R^1) \to M$ satisfying the following properties:

(i) $x(E^c) = 1 - x(E)$ for every $E \in B(R^1)$;

(ii) if $\{E_n\}_{n=1}^{\infty} \subset B(R^1)$, then $x(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} x(E_n)$;

where $B(R^1)$ is the Borel σ-algebra of the real line R^1, and E^c denotes the complement of the set E in R^1.

For example, if a is a fuzzy set from M, then the mapping x_a defined via

$$x_a(E) = \begin{cases} a \cap a^{\perp} & \text{if } 0,1 \notin E \\ a^{\perp} & \text{if } 0 \in E,\ 1 \notin E \\ a & \text{if } 0 \notin E,\ 1 \in E \\ a \cup a^{\perp} & \text{if } 0,1 \in E \end{cases} \qquad (1)$$

for any $E \in B(R^1)$ is an F-observable of (X,M) called the indicater of the fuzzy set a.

Especialy, the null F-observable of (X,M) is a mapping $\sigma: B(R^1) \to M$ such that

$$\sigma(E) = \begin{cases} [0]_X & \text{if } 0 \notin E \\ [1]_X & \text{if } 0 \in E. \end{cases} \qquad (E \in B(R^1)) \qquad (2)$$

If $f: R^1 \to R^1$ is a Borel measurable function, then $f \circ x: E \to x(f^{-1}(E))$, $E \in B(R^1)$ is an F-observable of (X,M). For example, if $f(t) = t^2$, $t \in R^1$, then by x^2 we mean $f \circ x$, etc.

Since (Dvurečenskij, A., Tirpáková, A. [3]) there is an one-to-one correspodence between an F-observable x and the system $\{B_x(t) := x((-\infty,t)) : t \in R^1\}$, in the papers [3,4], the sum of any pair x and y of F-observables of (X,M) has been introduced:

DEFINITION 3. By the sum of any pair of two F-observables x and y we mean a unique F-observable x + y for which we have

$$B_{x+y}(t) = \bigcup_{r\in Q}(B_x(r)\cap B_y(t-r)),\quad t\in R^1. \qquad (3)$$

In (Dvurečenskij, A., Tirpáková, A. [3,4]), there has been proved that x+y exists always, and it coincides with poitwisely defined sum of observables for σ-algebra of crisp subsets. Moreover, (i) x + y = y + x; (ii) (x + y) + z = = x + (y + z). The difference of x and y is defined as $x - y = x + (-y)$, where $(-y)(E) = y(\{t:\ -t\in E\})$, $E\in B(R^1)$.

If x is an F-observable and m is an F-state, then the mean value of x in m we shall understand the expression

$$m(x) = \int_{R^1} t\, dm_x(t) := \int x\, dm, \qquad (4)$$

if the integral exists and is finite, where m_x is a probability measure on $B(R^1)$ defined via $m_x(E) = m(x(E))$, $E\in B(R^1)$.

DEFINITION 4. We say, that a sequence $\{x_n\}_{n=1}^{\infty}$ of F-observables of a fuzzy quantum space (X,M) converges to an F-observable x almost everywhere in an F-state m (in short $x_n \longrightarrow x$ a.e. [m]), if for every $\varepsilon > 0$,

$$m\Big(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}((x_n - x)([-\varepsilon,\varepsilon]))\Big) = 1.$$

A mapping $\tau: M \longrightarrow M$ such that

(i) $\tau(a^\perp) = \tau(a)^\perp$, $a\in M$;

(ii) $\tau\Big(\bigcup_{i=1}^{\infty} a_i\Big) = \bigcup_{i=1}^{\infty}\tau(a_i)$, $\{a_i\}_{i=1}^{\infty}\subset M$

is called a homomorphism of (X,M). We say, that a homomorphism h of (X,M) is invariant in an F-state if

$$m(\tau(a)) = m(a),\quad a\in M.$$

A homomorphosm h of (X,M) invariant in an F-state m is said to be ergodic in m if the statement $m(a \wedge \tau(a^{\perp})) = 0 = m(\tau(a) \wedge a^{\perp})$ implies $m(a) \in \{0,1\}$. If τ is a homomorphism and x is an F-observable, then $\tau \circ x$: $E \longrightarrow (x(E))$, $E \in B(R^1)$, is an F-observable of (X,M).

ERGODIC THEOREMS

The calculus for observables that has been outlined in (Dvurečenskij, A., Tirpáková, A. [4]) enables us to formulate and prove the main ergodic theorems.

THEOREM 1. (Birkhoff's individual ergodic theorem). Let x be an F-observable of (X,M) and let τ be a homomorphism of (X,M) ergodic in an F-state m. Suppose $m(x) = 0$. Then

$$\frac{1}{n} \sum_{i=1}^{n-1} \tau^i \circ x \longrightarrow \sigma \quad \text{a.e.}[M]. \tag{5}$$

PROOF. Let us define $I_m = \{a \in M: m(a) = 0\}$, then a relation "$\sim$" defined via $a \sim b$ iff $m(a \wedge b^{\perp}) = 0 = m(a^{\perp} \wedge b)$ is the congruence, and moreover, $M/I_m = \{\bar{a} = \{b \in M: b \sim a\}: a \in M\}$ is the Boolean σ-algebra (in the sense of Sikorski [5]), where the complementation "$*$" in M/I_m is defined via $\bar{a}' = \overline{a^{\perp}}$, $a \in M$ and $\bigvee_i \bar{a}_i = \overline{\bigcup_i a_i}$, $\{a_i\} \subset M$. The mapping $h: M \longrightarrow M/I_m$ defined via $h(a) = \bar{a}$, $a \in M$ is a homomorphism from M onto M/I_m. The mapping $\mu: M/I_m \longrightarrow [0,1]$ for which $\mu(\bar{a}) = m(a)$, $a \in M$, is a probability measure on M/I_m.

Define a mapping $\bar{\tau}: M/I_m \longrightarrow M/I_m$ as follows

$$\bar{\tau}(\bar{a}) = \overline{\tau a}, \quad a \in M. \tag{6}$$

Then due to the invariancy of τ in m, $\bar{\tau}$ is a well-defined homomorphism of M/I_m, that is,

(i) $\bar{\tau}(\bar{0}) = \bar{0}$;
(ii) $\bar{\tau}(\bar{a}^{*}) = (\bar{\tau}(\bar{a}))^{*}$, $a \in M$;
(iii) $\bar{\tau}(\bigvee_{i=1}^{\infty} \bar{a}_i) = \bigvee_{i=1}^{\infty} \bar{\tau}(\bar{a}_i)$, $\{a_i\} \subset M$.

Moreover, $\bar{\tau}$ is invariant in μ, i.e., $\mu(\bar{\tau}(\bar{a})) = \mu(\bar{a})$, $a \in M$.

Let x be an F-observable of (X,M). Then $y := h \circ x$ is an observable of M/I_m. We recall that it means that

(i) $y(\emptyset) = \bar{0}$;

(ii) $y(E^c) = y(E)^\perp$, $E \in B(R^1)$;

(iii) $y(\bigcup_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} y(E_i)$, $\{E_i\} \subset B(R^1)$.

In the following we need the next lemma.

LEMMA 1. If τ is invariant in m, then for any $n = 0, 1, \dots$, we have

(i) $\bar{\tau}^n \circ y = h \circ \tau^n \circ x$. (7)

(ii) Let A be the minimal Boolean sub-σ-algebra of M/I_m containing all ranges of $\bar{\tau}^n \circ y$, $n = 1, 2, \dots$. Then $\bar{\tau}\bar{a} \in A$ for any $\bar{a} \in A$.

PROOF. (i) It is evident. (ii) Denote by $A_0 = \{\bar{a} \in A: \bar{\tau}\bar{a} \in A\}$. Then $\bar{0}, \bar{1} \in A_0$ and A_0 is a Boolean sub-σ-algebra of M/I_m containing all ranges of $\bar{\tau}^n \circ y$, $n \geq 0$. Hence $A_0 = A$.

Q.E.D.

<u>Continuation of the proof of Theorem 1</u>. It is evident that the Boolean sub-σ-algebra A in Lemma 1 has a countable generator. Due to (Varadarajan, V. S. [6], Theorem 1.4), there is an observable $z: B(R^1) \longrightarrow M/I_m$ such that

$\{z(E): E \in B(R^1)\} = A$,

and, moreover, there is a sequence of real-valued Borel functions $\{f_n\}_{n=1}^{\infty}$ such that

$(\bar{\tau}^n \circ y)(E) = z(f_n^{-1}(E))$, $E \in B(R^1)$, $n = 0, 1, \dots$, (8)

and f_n is essentially unique in the sense that if $z(f_n^{-1}(E)) = z(g_n^{-1}(E))$, $E \in B(R^1)$, then $z(\{t: f_n(t) \neq g_n(t)\}) = \bar{0}$.

From the construction of z it follows that $\bar{\tau}$ is z-measurable, that is, $\bar{\tau}(z(B(R^1))) \leq z(B(R^1))$. Due to (Dvurečenskij, A., Riečan, B. [7]), it is possible iff there is a Borel measurable transformation $T: R^1 \longrightarrow R^1$ such that

$\bar{\tau}(z(E)) = z(T^{-1}(E))$, $E \in B(R^1)$. (9)

Therefore,

$$\tau^n(z(E)) = z(T^{-n}(E)), \quad E \in B(R^1), \tag{10}$$

and

$$\tau_o^n y(E) = \tau^n(z(f^{-1}(E)) = z(T^{-n}(f_0^{-1}(E))) =$$

$$z((f_0 \circ T^n)^{-1}(E)) = z(f_n^{-1}(E)).$$

Due to (Varadarajan, V. S. [6], Theorem 1.4), we may assume without loss of generality that $f_n = f \circ T^n$, $n = 0, 1, \ldots$, for some Borel function f.

For observables in M/I_m there is a well-known (Varadarajan, V. S. [6]) way of definition of their sum, and the convergence almost everywhere of observables in M/I_m is same as for F-observables. Therefore,

$$\frac{1}{n}\sum_{i=1}^{n-1} \tau_o^n x \to \sigma \,[m] \quad \text{iff} \quad \frac{1}{n}\sum_{i=1}^{n-1} \tau_o^n y \to \sigma \quad \text{a.e.}\,[\mu],$$

where $\sigma(E) = \bar{0}$ if $0 \notin E$ and $\sigma(E) = \bar{1}$ otherwise.

The latest convergence is true iff

$$\frac{1}{n}\sum_{i=1}^{n-1} f(T^i) \circ z \to \sigma \quad \text{a.e. } [\mu], \text{ which is possible iff}$$

$$\frac{1}{n}\sum_{i=1}^{n-1} f(T^i(t)) \to 0 \quad \text{a.e.}[\mu_z], \text{ where } \mu_z(E) = \mu(z(E)),$$

$E \in B(R^1)$, is a probability measure on $B(R^1)$.

On the other hand,

$$m(x) = \int_{R^1} t \, dm_x(t) = \int_{R^1} t \, d\mu_y(t) = \int_{R^1} f(t) \, d\mu_z(t) = 0,$$

where $\mu_y(E) = \mu(y(E))$, $E \in B(R^1)$.

Take into account a dynamic system $(R^1, B(R^1), \mu_z, T)$. Then T is μ_z-invariant and ergodic in μ_z, i. e., (i) $\mu_z(T^{-1}(E)) = \mu_z(E)$, $E \in B(R^1)$; (ii) $T^{-1}(E) = E$ implies $\mu_z(E) \in \{0, 1\}$.

Therefore, due to (Halmos, P. R. [8]), for f the individual ergodic theorem holds, consequently (5) is proved.

Q.E.D.

REMARK. The individual ergodic theorem for fuzzy quantum spaces has been proved in (Harman, B., Riečan, B. [9]) but only under the restriction $\tau(x(B(R^1))) \leq x(B(R^1))$. Theorem 1 presents the most general case.

The <u>product</u> of two observables x and y is defined as follows

$$x_\circ y = ((x + y)^2 - x^2 - y^2)/2. \tag{11}$$

Hence, if a is a fuzzy set from M, then an <u>indefinite integral</u> of x is the expression

$$\int_a x \, dm := \int x_\circ x_a \, dm. \tag{12}$$

THEOREM 2. (<u>Maximal ergodic theorem</u>). Let τ be a homomorphism of a fuzzy quantum space (X,M) invariant in an F-state m. Let x be an F-observable of (X,M) which has a finite mean value in m. Let

$$S_k = \sum_{i=1}^{n-1} \tau^i \circ x, \; k = 1, \dots, n, \; a = \bigcup_{i=1}^{n} S_i((0,\infty)). \text{ Then}$$

$$\int_a x \, dm \geqslant 0. \tag{13}$$

PROOF. Let h be a mapping from the proof of Theorem 1. Then $h\circ(x + y) = h\circ x + h\circ y$, $h(f\circ x) = f\circ h(x)$, $h\circ(x_\circ y) = h\circ x_\circ h\circ y$. Therefore, if $x_{\bar a}$ is a question observable of M/I_m, that is, $x_{\bar a}(\{0\}) = \bar a^*$, $x_{\bar a}(\{1\}) = \bar a$, then we have (see the proof of Theorem 1) $\bar a = h(a) = \bigvee_{i=1}^{n} h(S_i((0,\infty))) = \bigvee_{i=1}^{n} z(s^{-1}((0,\infty))) = z(\bigcup_{i=1}^{n} s^{-1}((0,\infty))) = z(\max(0, s_1, \dots s_n) > 0)$,

where

$$s_k(t) = \sum_{i=1}^{k-1} f(T^i(t)), \; t \in R^1.$$

Hence,

$$\int_a x \, dm = \int x_\circ x_a \, dm = \int h\circ x_\circ h\circ x_a \, d\mu = \int h\circ(x_\circ x_{\bar a}) \, d\mu =$$
$$= \int_{\bar a} h\circ x \, d\mu = \int_A f(t) \, d\mu_z(t),$$ where $A = \{t \in R^1 : \max(0, s_1, \dots, s_n) > 0\}$. Applying maximal ergodic theorem for the dynamic system $(R^1, B(R^1), T, \mu_z)$, we see that $\int_A f(t) \, d\mu_z(t) \geqslant 0$.

Q.E.D.

Finally we prove the Poincaré recurrence theorems. We recall that for any two fuzzy sets $a, b \in M$ we define $a - b = a \cap b^{\perp}$. The fuzzy sets $a, b \in M$ are <u>orthogonal</u>, and we write $a \perp b$, if $a \leq b^{\perp}$.

THEOREM 3. (<u>Recurrence theorem</u>). Let τ be a homomorphism of a fuzzy quantum space (X,M) and let τ be invariant in an F-state. Then for all $a \in M$ we have

$$m(a - \bigcup_{j=1}^{\infty} \tau^j a) = 0.$$

PROOF. Let $b = a - \bigcup_{j=1}^{\infty} \tau^j a$, then $\{\tau^j b\}_{j=1}^{\infty}$ are orthogonal elements of M therefore $m(\bigcup_{j=1}^{\infty} \tau^j b) = \bigcup_{j=1}^{\infty} m(\tau^j b) = \bigcup_{j=1}^{\infty} m(b) < 1$. Hence $m(b) = 0$.

Q.E.D.

For $\{a_n\} \subset M$ we define $\limsup a_n = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} a_j$.

THEOREM 4. (<u>Strong recurrence theorem</u>). Let (X,M) be a fuzzy quantum space and τ be a homomorphism invariant in an F-state m. Then for all $a \in M$ we have

$$m(a - \limsup \tau^j a) = 0.$$

PROOF. Let $b = a - \limsup \tau^j a$, then $b = a \wedge \bigcup_{n=1}^{\infty} (\bigcup_{j=n}^{\infty} \tau^j a) = \bigcup_{n=1}^{\infty} (a \wedge (\bigcup_{j=n}^{\infty} \tau^j a)) = \bigcup_{n=1}^{\infty} (a - \bigcup_{j=n}^{\infty} \tau^j a) = \bigcup_{n=1}^{\infty} b_n$ where $b_n = a - \bigcup_{j=n}^{\infty} \tau^j a$, $n = 1, 2, \ldots$. Applying Theorem 3 to a map $T = \tau^n$ we get for $b_n^* = a - \bigcup_{j=1}^{\infty} T^j a$, $m(b_n^*) = 0$. But $b_n \leq b_n^*$, therefore, $m(b_n) = 0$, $n = 1, 2, \ldots$ and $m(b) = \lim_n m(b_n) = 0$.

Q.E.D.

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