PROBABILITY MEASURES IN THE FUZZY SET APPROACH TO QUANTUM LOGICS

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ABSTRACT. Probability measures on quantum logics yield the possibility of translating quantum logic notions into the language of fuzzy set theory. The analogies between set-theoretic operations and Giles bold union and intersection are used to give further justification of the Mączyński Orthogonality Postulate or the fifth axiom of Mackey.

The idea of a fuzzy set, invented by L.A. Zadeh [1] 23 years ago, is based on the notion of a membership function, the values of which belong to the unit interval $[0,1]$. However, functions, the values of which belong to the unit interval, are often met in pure and applied mathematics. Therefore it can be expected that in some such cases traditional mathematical notions can be expressed in the language of fuzzy set theory. In particular such hope can be cherished when we deal with probability measures or probability distributions. In this paper we shall show that this kind of approach is successful in the case of probability measures on quantum logics.

Def.1. By a **quantum logic** (or simply a **logic**) throughout this paper we mean partially ordered, orthocomplemented, σ-orthocomplete orthomodular set, i.e. a partially ordered set $\mathbf{L}$ in which

(i) the least element $\emptyset$ and the greatest element $\mathbf{I}$ exist,

(ii) the orthocomplementation map $' : \mathbf{L} \to \mathbf{L}$, such that $a'' = a$, $a \vee a' = \mathbf{I}$ and $a \leq b \Rightarrow b' \leq a'$ is admitted,

(iii) the least upper bound $\bigvee_i a_i$ of any sequence of elements $a_1, a_2, \ldots$ such that $a_i \leq a_j'$ for $i \neq j$ exists, and

(iv) the orthomodular identity $a \leq b \Rightarrow b = a \vee (a' \wedge b)$ holds.

Def.2. By a **probability measure** on a logic $\mathbf{L}$ we mean a map $m : \mathbf{L} \to [0,1]$ such that $m(\mathbf{I})=1$ and $m(\bigvee_i a_i) = \sum_i m(a_i)$ for any sequence of elements such that $a_i \leq a_j'$ for $i \neq j$. A set $\mathbf{S}$ of probability measures on $\mathbf{L}$ is called **full** iff $m(a) \leq m(b)$ for all $m \in \mathbf{S}$ implies $a \leq b$.

Elements of a logic, usually called **propositions**, and probability measures on a logic, usually called **states** form "dual" pairs such that to each pair $(\mathbf{a},m) \in \mathbf{L} \times \mathbf{S}$ a real number $m(\mathbf{a}) \in [0,1]$ is attached. This number is usually interpreted as the probability of obtaining a positive result in an experiment testing proposition **a** when a physical system is in the state represented by m. Despite the fact that the role of propositions and states in these pairs is symmetric, states are traditionally viewed as functions defined on propositions. However, the opposite point of view is also sometimes expressed [2,3] and it is in fact the basis of the approach developed by M.J. Mączyński [4,5], who proved the following theorem :

Theorem 1. (Mączyński [5], proof in [4])

(i) If **L** is a logic with a full set of probability measures **S**, then each $\mathbf{a} \in \mathbf{L}$ induces a function $\underline{a} : \mathbf{S} \to [0,1]$ where $\underline{a}(m) = m(\mathbf{a})$ for all $m \in \mathbf{S}$. The set of all such functions $\underline{L} = \{\underline{a} : \mathbf{a} \in \mathbf{L}\}$ satisfies the following condition

(OP) if $\underline{a}_1, \underline{a}_2, \ldots$ is a sequence of functions such that $\underline{a}_i + \underline{a}_j \leqslant 1$ for $i \neq j$, then there exists $\underline{b} \in \underline{L}$ such that $\underline{b} + \underline{a}_1 + \underline{a}_2 + \ldots = 1$.

$\underline{L}$ equipped with the natural partial order : $\underline{a} \leqslant \underline{b}$ iff $\underline{a}(m) \leqslant \underline{b}(m)$ for all $m \in \mathbf{S}$ and complementation $\underline{a}' = 1 - \underline{a}$ is isomorphic to **L**.

(ii) Conversely, if $\underline{L} \subseteq [0,1]^X$ is a set of functions in which the condition (OP) is satisfied, then it is a logic with respect to natural partial order $\leqslant$ and complementation. Every point $x \in X$ induces a probability measure m_x on $\underline{L}$ where $m_x(\underline{a}) = \underline{a}(x)$ for all $\underline{a} \in \underline{L}$ and the set of all such measures $\{m_x : x \in X\}$ is full.

The condition (OP) is called by Mączyński the **Orthogonality Postulate** and it is essential in the part (ii) of Theorem 1. In fact it replaces the fifth axiom from the well-known axiomatic approach of Mackey [6] and it is sufficient for the conditions (i),(iii) and (iv) of Definition 1 to be fulfilled.

Theorem 1 allows us to express quantum logic notions in the language of fuzzy set theory. Actually, since functions $\underline{a} : \mathbf{S} \to [0,1]$ described in part (i) of Theorem 1 can be interpreted as membership functions of fuzzy subsets of the set of probability measures **S**, we see that elements of the logic **L** are in one-to-one correspondence with fuzzy subsets of **S**. Moreover, natural partial order and complementation $\underline{a}' = 1 - \underline{a}$ in the set of membership functions are nothing else than standard fuzzy set inclusion and complementation as defined already by Zadeh [1]. Two fuzzy sets A,B which membership functions μ_A and μ_B satisfy the condition

$$\sup(0, \mu_A + \mu_B - 1) = 0 \quad \text{(or, equivalently, } \mu_A + \mu_B \leqslant 1) \qquad (1)$$

are called **weakly disjoint** sets and denoted $A \perp B$ by Giles [7] (or **W-separated** sets by Piasecki [8]). If the sum of membership functions of a sequence of fuzzy sets is not greater than 1, it coincides with the membership function of a fuzzy set obtained by the operation called **bold union** by Giles [7] :

$$\mu_{A \boldsymbol{\cup} B}(x) = \inf(1, \mu_A(x) + \mu_B(x)\ ). \qquad (2)$$

Therefore, Theorem 1 can be translated into the fuzzy set language in the following way (cf. [9]).

Theorem 1´. (i) Any logic **L** with a full set of probability measures **S** is isomorphic to a family $\mathcal{L}$ of fuzzy subsets of **S** equipped with the standard fuzzy set inclusion and complementation and satisfying the following **Fuzzy Orthogonality Postulate** :

(FOP) if $A_1, A_2, \ldots$ is a sequence of pairwise weakly disjoint sets, then $\sum_i \mu_{A_i} \leqslant 1$ and the fuzzy complement of the bold union $\boldsymbol{\cup}_i A_i$ exists in $\mathcal{L}$.

(ii) Conversly, any family $\mathcal{L}$ of fuzzy subsets of an universum X in which FOP is satisfied is a logic with respect to standard fuzzy set inclusion and complementation. Each point x of the universum X induces a probability measure m_x on $\mathcal{L}$ where $m_x(A) = \mu_A(x)$ for all $A \in \mathcal{L}$ and the set of all such measures is full.

If $\mathbf{a} \in \mathbf{L}$ and $m \in \mathbf{S}$, then the number $\underline{a}(m)$ can be interpreted in the language of fuzzy set theory as the grade of membership of a state m to a fuzzy subset A of the universum **S** determined by the property : "the outcome of an experiment testing the proposition **a** is positive" [9].

Let us now restrict ourselves to set **P** of **pure** probability measures on a logic **L** i.e. such measures which cannot be represented in the form of convex combinations of other measures and, therefore, represent pure states of a physical system. It is a well-known fact that pure states of classical systems are **dispersion-free**, i.e. the probability m(**a**) of obtaining a positive result in an experiment testing **a** when a system is in a pure state m is either 0 or 1. Consequently, any logic of a classical system regarded as a family of subsets of an universum **P** consists exclusively of crisp subsets of **P** while in the case of logics of quantum systems utilization of genuine fuzzy subsets of **P** cannot be avoided [9].

The other feature which distinguishes classical mechanics from quantum mechanics is that logics of classical systems regarded as partially ordered orthocomplemented sets are σ-complete Boolean algebras. When regarded as families of subsets of **P** they are simply Boolean algebras of crisp subsets of **P** with respect to standard set-theoretic inclusion and complementation. In the case of logics of quantum systems the Boolean type of logics cannot be expected. However, formal similarities between properties of set-theoretic complement, union and intersection and properties of Zadeh's standard fuzzy set complement and Giles' bold union (2) and intersection

$$\mu_{A \cap B}(x) = \sup(0, \mu_A(x) + \mu_B(x) - 1) \tag{3}$$

can be used to justify in another way the Mączyński Orthogonality Postulate or the fifth axiom of Mackey.

First of all let us notice that despite the fact that any σ-complete Boolean algebra of crisp sets is closed under countable unions of sequences of arbitrary sets, in the definition of a (probability) measure on a Boolean algebra of crisp sets only sequences of disjoint sets are taken into account. Weak disjointness of fuzzy sets was originally defined by Giles [7] by the following formula

$$A \perp B \quad \text{iff} \quad A \cap B = \emptyset \tag{4}$$

which is formally identical with the definition of disjoint crisp sets :

$$A \cap B = \emptyset. \tag{5}$$

Thus, the closedness of σ-complete Boolean algebra of crisp subsets of **P** under the set-theoretic union and complementation of disjoint crisp sets immediately gives the Fuzzy Orthogonality Postulate when we pass to fuzzy subsets of **P**. To achieve this we replace crisp subsets by fuzzy subsets, set-theoretic complementation by standard fuzzy set complementation and set-theoretic union and intersection of disjoint crisp sets by Giles' bold union and intersection of fuzzy sets.

It is noteworthy that standard fuzzy set union and intersection would not do the job since their properties imply that the standard fuzzy set complementation cannot be an orthocomplementation in the family of fuzzy sets partially ordered by the fuzzy set inclusion [10].

Finally, let us mention that the attempt at justifying FOP with the aid of the "quantization" mapping "q" from Boolean algebra of crisp subsets of **P** onto the family of fuzzy subsets of **P** proposed in [9] does not seem to be satisfactory. Besides the fact that assumptions about "q" adopted in [9] are too strong and force "q" to be an isomorphism (which it obviously should not be), according to our present opinion the direction of the desired mapping should be the opposite. Indeed, as has been pointed out, for example by Schiff [11], the passage from classical to quantum mechanics can be made in many different ways by adding to the classical equation of motion different terms which vanish when Planck's constant "h" diverges to 0. Therefore, the "quantization" mapping cannot be expected to be uniquely defined. On the contrary, the passage from quantum mechanics to classical mechanics, i.e. the "classical limit" is a unique operation and this (of course purely theoretical) operation can be applied even in the case of a single physical system with a well defined set of pure states **P**. The possibility of using this operation to study connections between the classical and quantum logics of physical systems will be studied in a forthcoming paper.

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