WELL-POSED FUZZY EXTENSIONS OF ILL-POSED LINEAR EQUALITY SYSTEMS*

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ABSTRACT. Linear equality systems with fuzzy parameters and crisp variables defined by the Zadeh's extension principle are called possibilistic linear equality systems. This study focuses on the problem of stability (with respect to small changes in the membership function of fuzzy parameters) of the solution in these systems.

1. INTRODUCTION.

Using mathematical models of reality we often encounter the problem of finding all solutions of the linear equality system

$$a_{i1}x_1 + \dots + a_{in}x_n = b_i \ , \qquad i = 1,\dots,m, \tag{1}$$

Such systems are frequent e.g. in operations research. Unfortunately, this problem is generally ill-posed; a small error of measurement of parameters, a_{ij}, b_i, or rounding error in digital computation may produce a large variation in the computed answer. In this paper the system (1) with fuzzy parameters a_{ij}, b_i, crisp variables x_j and extended relation is considered. We shall prove that the system (1) with continuous fuzzy numbers is well-posed; the solution depends continuously on the (fuzzy) data.

2. BASIC NOTATION AND DEFINITIONS.

DEFINITION 2.1. *A fuzzy number is a fuzzy set* $\tilde{a}$, $\tilde{a}: \mathbf{R} \to [0,1] = I$,

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which is upper-semicontinuous, normal, fuzzy convex and compactly supported.

We denote the set of all fuzzy numbers by $\mathcal{F}$ and the set of all continuous fuzzy numbers by $\mathcal{FC}$.

For any $\theta \geqq 0$ and $\tilde{a} \in \mathcal{FC}$ we now define the quantity

$$\omega(\tilde{a},\theta) = \max |\tilde{a}(u) - \tilde{a}(v)|$$

where the maximum is taken with respect to all points u,v in **R** such that $|u-v| \leqq \theta$. We shall call $\omega(\theta)$ the *modolus of continuity* of the fuzzy number $\tilde{a}$. The following statements hold [5]

(i) If $0 \leqq \theta \leqq \theta'$, then $\omega(\tilde{a},\theta) \leqq \omega(\tilde{a},\theta')$. (2)

(ii) If $\alpha > 0$, $\beta > 0$, then $\omega(\tilde{a},\alpha + \beta) \leqq \omega(\tilde{a},\alpha) + \omega(\tilde{a},\beta)$. (3)

(iii) $\lim\limits_{\theta\to 0} \omega(\tilde{a},\theta) = 0$. (4)

If $\tilde{a},\tilde{b} \in \mathcal{F}$ and $\lambda \in \mathbf{R}$ then $\tilde{a} + \tilde{b}$, $\tilde{a} - \tilde{b}$, $\lambda\tilde{a}$ are defined by the Zadeh's extension principle in the usual way. Denote by $[\tilde{a}]^{\alpha}$ the α-level set of $\tilde{a}$.

Let $\tilde{a},\tilde{b} \in \mathcal{F}$ and $[\tilde{a}]^{\alpha} = [a_1(\alpha), a_2(\alpha)]$, $[\tilde{b}]^{\alpha} = [b_1(\alpha), b_2(\alpha)]$.
We metricize $\mathcal{F}$ by the metric [6]

$$D(\tilde{a},\tilde{b}) = \sup_{\alpha\in I} \max_{i=1,2} \{|a_i(\alpha) - b_i(\alpha)|\}.$$

The truth value of the assertion "$\tilde{a}$ is equal to $\tilde{b}$",which we write $\tilde{a} = \tilde{b}$,is $\mathrm{Poss}(\tilde{a} = \tilde{b})$ defined in [9] as

$$\mathrm{Poss}(\tilde{a} = \tilde{b}) = \sup_{x\in\mathbf{R}} \min\{\tilde{a}(x),\tilde{b}(x)\} = (\tilde{a} - \tilde{b})(0). \qquad (5)$$

3.AUXILIARY PROPOSITIONS.

LEMMA 3.1.[6] *Let* $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d} \in \mathcal{F}$. *Then*

(i) $D(\lambda\tilde{a}, \lambda\tilde{b}) = |\lambda|\, D(\tilde{a}, \tilde{b})$ for $\lambda \in \mathbf{R}$,

(ii) $D(\tilde{a} + \tilde{c}, \tilde{b} + \tilde{d}) \leqq D(\tilde{a}, \tilde{b}) + D(\tilde{c}, \tilde{d})$,

(iii) $D(\tilde{a} - \tilde{c}, \tilde{b} - \tilde{d}) \leqq D(\tilde{a}, \tilde{b}) + D(\tilde{c}, \tilde{d})$.

LEMMA 3.2. *Let* $\lambda \neq 0$, $\mu \neq 0$ *be real numbers and let* $\tilde{a}$, $\tilde{b} \in \mathcal{FE}$ *be fuzzy numbers. Then*

$$\text{(i)} \quad \omega(\lambda\tilde{a},\theta) = \omega(\tilde{a}, \theta/|\lambda|), \tag{6}$$

$$\text{(ii)} \quad \omega(\lambda\tilde{a} + \mu\tilde{b},\theta) \leqq \omega\left(\frac{\theta}{|\lambda| + |\mu|}\right),$$

where $\omega(\theta) := \max\{\omega(\tilde{a},\theta), \omega(\tilde{b},\theta)\}$ for $\theta \geqq 0$.

LEMMA 3.3. *Let* $\delta \geqq 0$ *and* $\tilde{a},\tilde{b} \in \mathcal{FE}$. *If* $D(\tilde{a},\tilde{b}) \leqq \delta$, *then*

$$\sup_{t\in\mathbf{R}} |\tilde{a}(t) - \tilde{b}(t)| \leqq \max\{\omega(\tilde{a},\delta),\omega(\tilde{b},\delta)\}$$

The proof of this lemma is carried out analogously to the proof of Lemma 3.5.[4].

4.WELL-POSED FUZZY EXTENSIONS.

Generalizing the system (1) we consider the following possibilistic linear equality system

$$\tilde{a}_{i1}x_1 + \ldots + \tilde{a}_{in}x_n = \tilde{b}, \; i = 1,\ldots,m \tag{7}$$

where $\tilde{a}_{ij},\tilde{b}_i \in \mathcal{FE}$ and "=" denotes possibility.
We denote by $\mu_i(x)$ the degree of satisfaction of the i-th equation at the point x in (7),i.e.

$$\mu_i(x) = \mathrm{Poss}(\tilde{a}_{i1}x_1 + \ldots + \tilde{a}_{in}x_n = \tilde{b}).$$

Following Bellman-Zadeh [1] the solution (or the fuzzy set of weak feasible solutions [2]) of the system (7) is the intersection of the μ_i's such that $\mu(x)=\min_{i=1,..,m} \mu_i(x)$ for $x\in\mathbf{R}^n$. A measure of consistency for the system (7) is $\max_{x\in\mathbf{R}^n} \mu(x)=\mu^*$. An important question is the influence of perturbations of the data to the solution [10] of the system (7). In the sequel we will assume, that there is a collection of data $\tilde{a}^{\delta}_{ij},\tilde{b}^{\delta}_{i}$ available with the property

$$\max_{i,j}D(\tilde{a}_{ij},\tilde{a}^{\delta}_{ij}) \leqq \delta, \ \max_{i}D(\tilde{b}_{i},\tilde{b}^{\delta}_{i}) \leqq \delta. \tag{8}$$

Then we have to solve the following problem

$$\tilde{a}^{\delta}_{ij}x_1 + \dots + \tilde{a}^{\delta}_{in}x_n = \tilde{b}^{\delta}_{i}, \quad i=1,\dots,m \tag{9}$$

In the following theorem we establish a stability property (with respect to perturbations (8)) of the solution and the measure of consistency in possibilistic systems (7),(9).

THEOREM 4.1.*Let* $\delta \geqq 0$ *be a real number and let* $\tilde{a}_{ij},\tilde{b}_{i},\tilde{a}^{\delta}_{ij},\tilde{b}^{\delta}_{i}$ *be continuous fuzzy numbers.If* (8) *holds,then*

$$\sup_{x\in\mathbf{R}^n} |\mu(x) - \mu^{\delta}(x)| \leqq \omega(\delta), \tag{10}$$

$$|\mu^* - \mu^*(\delta)| \leqq \omega(\delta) \tag{11}$$

where $\omega(\delta)= \max_{i,j}\{\omega(\tilde{a}_{ij},\delta),\omega(\tilde{a}^{\delta}_{ij},\delta),\omega(\tilde{b}_{i},\delta),\omega(\tilde{b}^{\delta}_{i},\delta)\}$, μ^{δ} is the solution of perturbed problem (9) and $\mu^*(\delta)$ is the measure of consistency for the system (9).

*Proof.*It is sufficient to show that

$$|\mu_i(x) - \mu^{\delta}_i(x)| \leqq \omega(\delta) \tag{12}$$

for each $x\in\mathbf{R}^n$ and $1\leqq i\leqq m$, because (10) and (11) follow from (12). Let $x\in\mathbf{R}^n$ and $i\in\{1,\dots,m\}$ arbitrarily fixed.From (5) it follows that

$$\mu_i(x)=\left[\sum_{j=1}^{n}\tilde{a}_{ij}x_j - \tilde{b}_i\right](0),\quad \mu_i^\delta(x)=\left[\sum_{j=1}^{n}\tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta\right](0)$$

Applying Lemma 3.1.and (8) we have

$$D\left[\sum_{j=1}^{n}\tilde{a}_{ij}x_j - \tilde{b}_i,\ \sum_{j=1}^{n}\tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta\right] \leqq \sum_{j=1}^{n}|x_j|D(\tilde{a}_{ij},\tilde{a}_{ij}^\delta)$$

$$+ D(\tilde{b}_i,\tilde{b}_i^\delta) \leqq \delta(|x|_1+1)$$

where $|x|_1 = |x_1|+\ldots+|x_n|$. By Lemma 3.2.we get

$$\max\left\{\omega\left[\sum_{j=1}^{n}\tilde{a}_{ij}x_j - \tilde{b}_i,\delta\right],\omega\left[\sum_{j=1}^{n}\tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta,\delta\right]\right\} \leqq \omega\left[\frac{\delta}{|x|_1+1}\right].$$

Finally, applying Lemma 3.3. we have

$$|\mu_i(x)-\mu_i^\delta(x)| = \left|\left[\sum_{j=1}^{n}\tilde{a}_{ij}x_j - \tilde{b}_i\right](0) - \left[\sum_{j=1}^{n}\tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta\right](0)\right|$$

$$\leqq \sup_{t\in\mathbf{R}}\left|\left[\sum_{j=1}^{n}\tilde{a}_{ij}x_j - \tilde{b}_i\right](t)-\left[\sum_{j=1}^{n}\tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta\right](t)\right|$$

$$\leqq \omega\left[\frac{\delta(|x|_1+1)}{|x|_1+1}\right] = \omega(\delta).$$

Which proves the theorem.

REMARK 4.1. From (4),(10) and (11) it follows that $\sup_{x\in\mathbf{R}^n}|\mu(x)-\mu^\delta(x)| \to 0$, $|\mu^* - \mu^*(\delta)| \to 0$ as $\delta \to 0$, which means stability with respect to perturbations (8) of the solution and measures of consistency in possibilistic systems (7) and (9).

REMARK 4.2.As an immediate consequence of this theorem we obtain the following result [3]: If the fuzzy numbers in (7) and (9) satisfy the Lipschitz condition with constant $L > 0$, then $\sup_{x\in\mathbf{R}^n}|\mu(x) - \mu^\delta(x)| \leqq L\delta$, $|\mu^* - \mu^*(\delta)| \leqq L\delta$. Furthermore, similar estimations were obtained in the case of symmetrical trapezoidal fuzzy number parameters [8] and in the case of symmetrical triangular fuzzy number parameters [7].

REMARK 4.3.It is easily checked that in the general case $\tilde{a}_{ij},\tilde{b}_i\in\mathcal{F}$ the solution of the system (7) may be unstable under small variations in the membership function of fuzzy parameters.

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