On the Null-additivity and the Autocontinuity of Fuzzy Measure

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(Extended abstract)

Since Sugeno [1] introduced the concept of fuzzy measure, the structural characteristics of fuzzy measure have been studied in a lot of papers, such as [2], [3], [4] and [5]. These characteristics play important roles in the theory of fuzzy measure and fuzzy integral. In this paper, a further research on the null-additivity and the autocontinuity of fuzzy measure will be made.

1. Definitions

Let X be a nonempty set, $\mathcal{F}$ be a σ-algebra of subsets of X, and $\mu:\mathcal{F}\longrightarrow[0,\infty]$ be a nonnegative extended real-valued set function.

Definition 1. μ is called to be null-additive, if $\mu(E\cup F)=\mu(E)$ whenever $E\in\mathcal{F}$, $F\in\mathcal{F}$, $E\cap F=\phi$ and $\mu(E)=0$.

Definition 2. μ is called to be autocontinuous from above (resp. from below), if

$$\mu(A\cup B_n)\longrightarrow\mu(A)$$

$$(\text{resp. } \mu(A-B_n)\longrightarrow\mu(A)\ )$$

whenever $A\in\mathcal{F}$, $B_n\in\mathcal{F}$, $A\cap B_n=\phi$ (resp. $B_n\subset A$), $n=1,2,\cdots$, $\mu(B_n)\longrightarrow 0$; μ is called to be autocontinuous, if it is both autocontinuous from above and from below.

Definition 3. μ is called to be uniformly autocontinuous from above (resp. from below), if $\forall\varepsilon > 0$, $\exists\delta = \delta(\varepsilon) > 0$, such that

$$\mu(A) - \varepsilon \le \mu(A \cup B) \le \mu(A) + \varepsilon$$

$$(\text{resp. } \mu(A) - \varepsilon \le u(A - B) \le u(A) + \varepsilon)$$

whenever $A \in \mathcal{F}$, $B \in \mathcal{F}$, $A \cap B = \phi$ (resp. $B \subset A$), $\mu(B) \le \delta$; μ is called to be uniformly autocontinuous, if it is both uniformly autocontinuous from above and from below.

Definition 4. μ is a fuzzy measure, if

(1) $\mu(\phi) = 0$;

(2) $A \subset B \Longrightarrow \mu(A) \le \mu(B)$;

(3) $\{A_n\} \subset \mathcal{F}$, $A_1 \subset A_2 \subset \cdots \Longrightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;

(4) $\{A_n\} \subset \mathcal{F}$, $A_1 \supset A_2 \supset \cdots$, and $\exists n_0$, such that $\mu(A_{n_0}) < \infty \Longrightarrow$
$\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

2. Principal results

We know that the uniform autocontinuity from above (resp. from below) implies the autocontinuity from above(resp. from below), and the latter implies the null-additivity. When μ is a fuzzy measure, the uniform autocontinuity from above is equivalent to the uniform autocontinuity from below.

In the following, we always suppose μ is a fuzzy measure on the measurable space $(X, \mathcal{F})$.

Theorem 1. If μ is autocontinuous from above, then it is autocontinuous from below. When $\mu(X) < \infty$, the autocontinuity from below implies the autocontinuity from above, and therefore, the autocontinuity, the autocontinuity from above and the autocontinuity from below are equivalent.

There is an example (cf. [4]) to show that the autocontinuity from below does not imply the autocontinuity from above when $\mu(X)$

$= \infty$. This example shows that as well, in general, the null-additivity is not equivalent to the autocontinuity. However, we have the following result.

Theorem 2. If X is countable and $\mu(X) < \infty$, then the null-additivity is equivalent to the autocontinuity.

The following example shows that an autocontinuous fuzzy measure may be not uniformly autocontinuous.

Example. Let $X = \{1,2,\cdots\}$, $\mathcal{F}$ be the power set of X, $\mu(E) = \exp\{\sum_{i \in E} \frac{1}{i}\} - 1$, $\forall E \in \mathcal{F}$. μ is autocontinuous, and it is a quasi-measure (cf. [3]), but it is not uniformly autocontinuous.

There is an example showing that the autocontinuity is not equivalent to the uniform autocontinuity, even if μ is finite.

Theorem 3. Quasi-measure is autocontinuous, and finite quasi-measure is uniform autocontinuous.

The following results are useful to constructing some examples of fuzzy measure possessing the null-additivity, the autocontinuity or the uniform autocontinuity respectively.

Let μ_1, μ_2 be fuzzy measures. If we set $\mu(E) = \mu_1(E) + \mu_2(E)$, $\nu(E) = \mu_1(E) \cdot \mu_2(E)$, $\forall E \in \mathcal{F}$, then μ and ν are fuzzy measures as well.

Theorem 4. If both μ_1 and μ_2 are null-additive (resp. autocontinuous or uniformly autocontinuous), then so is μ.

Theorem 5. If both μ_1 and μ_2 are null-additive (resp. autocontinuous), then so is ν.

There are some examples showing that when $\nu(X) = \infty$, ν may be not uniformly autocontinuous, even if both μ_1 and μ_2 are uniformly autocontinuous.

However, we have

Theorem 6. If both μ_1 and μ_2 are finite and uniformly autocontinuous, then so is ν.

3. Open problems

(1) Givingup the restriction that X is countable, does the conclusion of Theorem 2 remain true?

(2) We have proved such a proposition: Let $\{E_n\}\subset\mathcal{F}$, $\mu(E_n)\longrightarrow 0$, if μ is autocontinuous from above (resp. from below, and $\mu(X)<\infty$), then there exists some subsequence $\{E_{n_i}\}$ of $\{E_n\}$, such that

$$\mu\left(\bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty}E_{n_i}\right)=0$$

$$\left(\text{resp. } \mu\left(\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}E_{n_i}^{c}\right)=\mu(X)\right).$$

But we don't know whether the condition $\mu(X)<\infty$ may be abandoned.

References

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