FUZZY INTEGRALS ON L-FUZZY SETS AND THEIR PROPERTIES

Qiao Zhong
Department of Basic Sciences,
Hebei Institute of Architectural Engineering,
Zhangjiakou, Hebei, China

Abstract

In this paper we consider the real-valued fuzzy measures and fuzzy integrals for L-fuzzy sets (where L is a pseudocomplemented infinitely distributive complete lattice), this is only a summary of our work. On a fuzzy σ-algebra of L-fuzzy sets, the concepts of the fuzzy measure and the fuzzy integral are introduced, some equivalent forms of fuzzy integrals on L-fuzzy sets are given. Moreover we discuss several properties of fuzzy integrals on L-fuzzy sets, and prove some convergence theorems for a sequence of fuzzy integrals on L-fuzzy sets.

Keywords: Lattice, L-fuzzy set, Fuzzy measure,
Fuzzy integral on L-fuzzy set.

1. Introduction

The fuzzy measures and fuzzy integrals for Zadeh's fuzzy sets were studied by Qiao and Wang[5,6,7,12] (where a fuzzy set is a mapping from a nonempty set X to the interval [o,1]). In this paper we shall establish a theory of the fuzzy measure and fuzzy integral on a fuzzy σ-algebra of L-fuzzy sets (where a L-fuzzy set is a mapping from X to the pseudocomplemented infinitely distributive complete lattice L).

Throughout this paper, L denote a pseudocomplemeted infinitely distributive complete lattice, namely the lattice L satisfies the following conditions:

(1) For any $H \subset L$, $\bigwedge_{h\in H} h$ and $\bigvee_{h\in H} h$ are existent in L;

(2) For any $H \subset L$, $a \in L$, then

$$a \wedge (\bigvee_{h\in H} h) = \bigvee_{h\in H} (a \wedge h), \quad a \vee (\bigwedge_{h\in H} h) = \bigwedge_{h\in H} (a \vee h) ;$$

(3) There exists a mapping N: $L \longrightarrow L$, such that

$N(N(a))=a$ for any $a\in L$, and if $a\leqslant b$, then $N(b)\leqslant N(a)$ for any $a,b\in L$. That is, N is a pseudo-complementation on L. Obviously, $N(I)=\theta$, $N(\theta)=I$, where I and θ are respectively the greatest and least element of L.

In this paper, X denote a nonempty set, $\mathfrak{F}_L(X)=\{\underset{\sim}{A};\ \underset{\sim}{A}: X\longrightarrow L\}$ is the class of all L-fuzzy sets on X, $\mathcal{P}_L(X)=\{\underset{\sim}{E};\ \underset{\sim}{E}: X\longrightarrow\{I,\theta\}\}$, evidently, $\mathcal{P}_L(X)\subset\mathfrak{F}_L(X)$. Thus $\mathfrak{F}_L(X)$ has a lattice structure induced pointwise by L, namely $\mathfrak{F}_L(X)$ is a pseudocomplemented infinitely distributive complete lattice. The greatest element of $\mathfrak{F}_L(X)$ is the L-fuzzy set X : $X(x)\equiv I$ for any $x\in X$. The least element of $\mathfrak{F}_L(X)$ is the L-fuzzy set $\underset{\sim}{\emptyset}$: $\underset{\sim}{\emptyset}(x)\equiv\theta$ for any $x\in X$.

For the class $\mathcal{P}_L(X)$, we observe the fact:

$$X,\underset{\sim}{\emptyset}\in\mathcal{P}_L(X),\ \text{and}\ X^c=\underset{\sim}{\emptyset}\ ,\ \underset{\sim}{\emptyset}^c=X;$$

$$\text{For any}\ \underset{\sim}{E}\in\mathcal{P}_L(X),\ \underset{\sim}{E}^c\cap\underset{\sim}{E}=\underset{\sim}{\emptyset}\ ,\ \underset{\sim}{E}^c\cup\underset{\sim}{E}=X.$$

We make the following conventions:

$$\bigcup_{t\in\emptyset}\{\cdot\}=\underset{\sim}{\emptyset},\quad \bigcap_{t\in\emptyset}\{\cdot\}=X,\quad \inf_{t\in\emptyset}\{a_t;\ a_t\in[0,\infty]\}=\infty\ ,\ 0\cdot\infty=0,$$

where $\emptyset$ is the classical empty set.

2. Fuzzy σ-Algebra and Fuzzy Measure for L-Fuzzy Sets

Definition2.1 A nonempty subclass $\underset{\sim}{\mathfrak{F}}$ of $\mathfrak{F}_L(X)$ is called a fuzzy σ-algebra, if it satisfies the following conditions:
(1) $\underset{\sim}{\emptyset}$, $X\in\underset{\sim}{\mathfrak{F}}$; (2) If $\underset{\sim}{A}\in\underset{\sim}{\mathfrak{F}}$, then $\underset{\sim}{A}^c\in\underset{\sim}{\mathfrak{F}}$; (3) If $\{\underset{\sim}{A}_n\}\subset\underset{\sim}{\mathfrak{F}}$, then $\bigcup_{n=1}^{\infty}\underset{\sim}{A}_n\in\underset{\sim}{\mathfrak{F}}$.

Evidently, $\mathfrak{F}_L(X)$ and $\mathcal{P}_L(X)$ are fuzzy σ-algebras. If $\underset{\sim}{\mathfrak{F}}$ is a fuzzy σ-algebra, then $\mathcal{B}=\mathcal{P}_L(X)\cap\underset{\sim}{\mathfrak{F}}$ is a fuzzy σ-algebra.

In this paper, $\underset{\sim}{\mathfrak{F}}$ will always denote a fuzzy σ-algebra.

Definition2.2 A mapping $\underset{\sim}{\mu}:\underset{\sim}{\mathfrak{F}}\longrightarrow[0,\infty]$ is said to be a fuzzy measure on $\underset{\sim}{\mathfrak{F}}$, if and only if

(1) $\underset{\sim}{\mu}(\underset{\sim}{\emptyset})=0$;

(2) For any $\underset{\sim}{A},\underset{\sim}{B}\in\underset{\sim}{\mathfrak{F}}$, if $\underset{\sim}{A}\subset\underset{\sim}{B}$, then $\underset{\sim}{\mu}(\underset{\sim}{A})\leqslant\underset{\sim}{\mu}(\underset{\sim}{B})$ (monotonicity);

(3) Whenever $\{\underset{\sim}{A}_n\}\subset\underset{\sim}{\mathfrak{F}}$, $\underset{\sim}{A}_n\subset\underset{\sim}{A}_{n+1}$, $n=1,2,\cdots$, then

$\underset{\sim}{\mu}(\bigcup_{n=1}^{\infty}\underset{\sim}{A}_n)=\lim_{n\to\infty}\underset{\sim}{\mu}(\underset{\sim}{A}_n)$ (continuity from below);

(4) Whenever $\{\underset{\sim}{A}_n\}\subset\underset{\sim}{\mathfrak{F}}$, $\underset{\sim}{A}_n\supset\underset{\sim}{A}_{n+1}$, $n=1,2,\cdots$, and there exists n_0 such that $\underset{\sim}{\mu}(\underset{\sim}{A}_{n_0})<\infty$, then $\underset{\sim}{\mu}(\bigcap_{n=1}^{\infty}\underset{\sim}{A}_n)=\lim_{n\to\infty}\underset{\sim}{\mu}(\underset{\sim}{A}_n)$ (continuity from

above). The triple $(X, \mathfrak{F}, \mu)$ is called a fuzzy measure space.

Definition2.3 The fuzzy measure μ is called autocontinuous from above (resp. from below), if $\mu(B_n) \longrightarrow 0$, then

$$\mu(A \cup B_n) \longrightarrow \mu(A) \quad (\text{resp. } \mu(A \cap B_n^c) \longrightarrow \mu(A)),$$

whenever $A \in \mathfrak{F}$, $\{B_n\} \subset \mathfrak{F}$. μ is called autocontinuous, if it is both autocontinuous from above and from below.

Definition2.4 Let $A \in \mathfrak{F}$, $\mu(A) < \infty$. The fuzzy measure μ is said to be pseudo-autocontinuous from above with respect to A (resp. from below with respect to A), if for any $\{B_n\} \subset \mathfrak{F}$, when $\mu(B_n \cap A) \longrightarrow \mu(A)$, we have

$$\mu((B_n^c \cap A) \cup E) \longrightarrow \mu(E) \quad (\text{resp. } \mu(B_n \cap E) \longrightarrow \mu(E))$$

whenever $E \in A \cap \mathfrak{F}$. μ is called pseudo-autocontinuous with respect to A, if it is both pseudo-autocontinuous from above with respect to A and from below with respect to A. (Where $A \cap \mathfrak{F} = \{A \cap D;\ D \in \mathfrak{F}\}$.)

Definition2.5 The fuzzy measure μ is called null-subtractive, if we have $\mu(A \cap B^c) = \mu(A)$ whenever $A, B \in \mathfrak{F}$, $\mu(B) = 0$.

Definition2.6 Let $A \in \mathfrak{F}$, $\mu(A) < \infty$. The fuzzy measure μ is said to be pseudo-null-subtractive with respect to A, if for any $E \in A \cap \mathfrak{F}$, we have $\mu(B \cap E) = \mu(E)$ whenever $B \in \mathfrak{F}$, $\mu(B \cap A) = \mu(A)$.

Proposition2.7 If the fuzzy measure μ is autocontinuous from below, then it is null-subtractive.

Proposition2.8 If the fuzzy measure μ is pseudo-autocontinuous from below with respect to A, then it is pseudo-null-subtractive with respect to A.

3. Fuzzy Integral on L-Fuzzy Set

Definition3.1 A mapping $f: X \longrightarrow (-\infty, \infty)$ is called a measurable function on $\mathfrak{F}$, if $F_\alpha \in \mathfrak{F}$ for every $\alpha \in (-\infty, \infty)$, where F_α is the L-fuzzy set such that $F_\alpha(x) = \begin{cases} I & \text{if } f(x) \geq \alpha \\ \theta & \text{if } f(x) < \alpha \end{cases}$ for any $x \in X$.

Denote: $M = \{f;\ f \text{ is a measurable function on } \mathfrak{F}\}$,

$M^+ = \{f;\ f \in M,\ f \geq 0\}$.

Definition3.2 Let $A \in \mathfrak{F}$, $f \in M^+$. The fuzzy integral of f on A with respect to μ is defined by $\int_A f d\mu = \sup_{\alpha \in [0, \infty]} (\alpha \wedge \mu(A \cap F_\alpha))$,

where F_α is the L-fuzzy set such that $F_\alpha(x) = \begin{cases} I & \text{if } f(x) \geq \alpha \\ \theta & \text{if } f(x) < \alpha \end{cases}$ $x \in X$.

Remark: When $L=[0,1]$, $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu}$ is the fuzzy integral defined in [6,7,12]. If we take $L=\{0,1\}$, then $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu}$ is the fuzzy integral in [9,11].

Theorem3.3 $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} = \sup_{\alpha\in[0,\infty)} [\alpha\wedge\underset{\sim}{\mu}(\underset{\sim}{A}\cap F_\alpha)] = \sup_{\alpha\in(0,\infty)} [\alpha\wedge\underset{\sim}{\mu}(\underset{\sim}{A}\cap F_\alpha)]$.

Theorem3.4 $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} = \sup_{\alpha\in[0,\infty)} [\alpha\wedge\underset{\sim}{\mu}(\underset{\sim}{A}\cap F_{\bar\alpha})] = \sup_{\alpha\in(0,\infty)} [\alpha\wedge\underset{\sim}{\mu}(\underset{\sim}{A}\cap F_{\bar\alpha})]$,

where $F_{\bar\alpha}$ is the L-fuzzy set such that $F_{\bar\alpha}(x)=\begin{cases} I & \text{if } f(x)>\alpha \\ \theta & \text{if } f(x)\leqslant\alpha \end{cases}$ $x\in X$.

Theorem3.5 $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} = \sup_{\underset{\sim}{E}\in\mathfrak{B}} [(\inf_{\underset{\sim}{E}(x)=I} f(x))\wedge\underset{\sim}{\mu}(\underset{\sim}{A}\cap\underset{\sim}{E})]$

$= \sup_{\underset{\sim}{E}\in\underset{\sim}{\mathfrak{F}}} [(\inf_{\underset{\sim}{E}(x)>\theta} f(x))\wedge\underset{\sim}{\mu}(\underset{\sim}{A}\cap\underset{\sim}{E})]$,

where $\mathfrak{B}=\wp_L(X)\cap\underset{\sim}{\mathfrak{F}}$.

Definition3.6 $s\in\underset{\sim}{M}^+$ is called a nonnegative simple function on $\underset{\sim}{\mathfrak{F}}$, if there exist $\underset{\sim}{E}_1,\cdots,\underset{\sim}{E}_n\in\mathfrak{B}$ (where $\mathfrak{B}=\wp_L(X)\cap\underset{\sim}{\mathfrak{F}}$, $\underset{\sim}{E}_i\neq\emptyset$, $i=1,2,\cdots,n$, $\underset{\sim}{E}_i\cap\underset{\sim}{E}_j=\emptyset$, $i\neq j$, $\bigcup_{i=1}^n\underset{\sim}{E}_i=X$) and real numbers $\alpha_1,\cdots,\alpha_n\in[0,\infty)$ (where $\alpha_i\neq\alpha_j$, $i\neq j$) such that for any $x\in X$,

$$s(x)=\alpha_i \quad \text{if } \underset{\sim}{E}_i(x)=I,\ i=1,2,\cdots,n.$$

Denote the set of all nonnegative simple functions on $\underset{\sim}{\mathfrak{F}}$ by H.

Proposition3.7 If $s\in H$ has two representations:

$$s(x)=\alpha_i \quad \text{if } \underset{\sim}{E}_i(x)=I,\ i=1,2,\cdots,n, \text{ for any } x\in X,$$

$$s(x)=\beta_s \quad \text{if } \underset{\sim}{G}_s(x)=I,\ s=1,2,\cdots,m, \text{ for any } x\in X,$$

(where $\underset{\sim}{E}_i$, $\underset{\sim}{G}_s$, α_i, β_s satisfy the condition given in Def.3.6) then $\{\alpha_1,\cdots,\alpha_n\}=\{\beta_1,\cdots,\beta_m\}$, and if $\alpha_i=\beta_s$, then $\underset{\sim}{E}_i=\underset{\sim}{G}_s$.

Theorem3.8 Let $\underset{\sim}{A}\in\underset{\sim}{\mathfrak{F}}$, $f\in\underset{\sim}{M}^+$. For any $s\in H$,

$$s(x)=\alpha_i \quad \text{if } \underset{\sim}{E}_i(x)=I,\ i=1,2,\cdots,n, \text{ for any } x\in X,$$

where $\underset{\sim}{E}_i$, α_i satisfy the condition given in Definition3.6, if we define $Q_{\underset{\sim}{A}}(s)=\bigvee_{i=1}^n [\alpha_i\wedge\underset{\sim}{\mu}(\underset{\sim}{A}\cap\underset{\sim}{E}_i)]$, then $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu}=\sup_{s\in H(f)} Q_{\underset{\sim}{A}}(s)$, where $H(f)=\{s;\ s\leqslant f,\ s\in H\}$.

4. Properties of Fuzzy Integrals on L-Fuzzy Sets

Proposition4.1 Let $\alpha,\beta\in[0,\infty)$, $\{\alpha_n\}\subset[0,\infty)$.

(1) $F_{\bar\alpha}\subset F_\alpha$, where $F_{\bar\alpha}(x)=\begin{cases} I & \text{if } f(x)>\alpha \\ \theta & \text{if } f(x)\leqslant\alpha \end{cases}$, $F_\alpha(x)=\begin{cases} I & \text{if } f(x)\geqslant\alpha \\ \theta & \text{if } f(x)<\alpha \end{cases}$ $x\in X$;

(2) If $\alpha \leqslant \beta$, then $F_\alpha \supset F_\beta$, $F_{\bar\alpha} \supset F_{\bar\beta}$;

(3) If $\alpha_n \nearrow \alpha$ and $\alpha_n < \alpha$, then $\bigcap_{n=1}^{\infty} F_{\alpha_n} = \bigcap_{n=1}^{\infty} F_{\bar\alpha_n} = F_\alpha$;

(4) If $\alpha_n \searrow \alpha$ and $\alpha_n > \alpha$, then $\bigcup_{n=1}^{\infty} F_{\alpha_n} = \bigcup_{n=1}^{\infty} F_{\bar\alpha_n} = F_{\bar\alpha}$.

<u>Theorem4.2</u> The fuzzy integrals on L-fuzzy sets have the following properties:(where $\underset{\sim}{A}, \underset{\sim}{B} \in \underset{\sim}{\mathcal{F}}$, f, $f_1, f_2 \in \underset{\sim}{M}^+$)

(1) If $\underset{\sim}{\mu}(\underset{\sim}{A})=o$, then $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu}=o$;

(2) If $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu}=o$, then $\underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\bar o})=o$;

(3) If $f_1 \leqslant f_2$, then $\int_{\underset{\sim}{A}} f_1 d\underset{\sim}{\mu} \leqslant \int_{\underset{\sim}{A}} f_2 d\underset{\sim}{\mu}$;

(4) If $\underset{\sim}{A} \subset \underset{\sim}{B}$, then $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} \leqslant \int_{\underset{\sim}{B}} fd\underset{\sim}{\mu}$;

(5) For any $a \in [o, \infty)$, $\int_{\underset{\sim}{A}} ad\underset{\sim}{\mu} = a \wedge \underset{\sim}{\mu}(\underset{\sim}{A})$;

(6) $\int_{\underset{\sim}{A}} (f_1 \vee f_2) d\underset{\sim}{\mu} \geqslant \int_{\underset{\sim}{A}} f_1 d\underset{\sim}{\mu} \vee \int_{\underset{\sim}{A}} f_2 d\underset{\sim}{\mu}$;

(7) $\int_{\underset{\sim}{A}} (f_1 \wedge f_2) d\underset{\sim}{\mu} \leqslant \int_{\underset{\sim}{A}} f_1 d\underset{\sim}{\mu} \wedge \int_{\underset{\sim}{A}} f_2 d\underset{\sim}{\mu}$;

(8) $\int_{\underset{\sim}{A} \cup \underset{\sim}{B}} fd\underset{\sim}{\mu} \geqslant \int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} \vee \int_{\underset{\sim}{B}} fd\underset{\sim}{\mu}$;

(9) $\int_{\underset{\sim}{A} \cap \underset{\sim}{B}} fd\underset{\sim}{\mu} \leqslant \int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} \wedge \int_{\underset{\sim}{B}} fd\underset{\sim}{\mu}$;

(10) $\int_{\underset{\sim}{A}} (f+a) d\underset{\sim}{\mu} \leqslant \int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} + \int_{\underset{\sim}{A}} ad\underset{\sim}{\mu}$, $a \in [o, \infty)$;

(11) For any $a \in [o, \infty)$, if $|f_1 - f_2| \leqslant a$, then $\left| \int_{\underset{\sim}{A}} f_1 d\underset{\sim}{\mu} - \int_{\underset{\sim}{A}} f_2 d\underset{\sim}{\mu} \right| \leqslant a$.

<u>Definition4.3</u> $f \in \underset{\sim}{M}^+$ is called fuzzy integrable on $\underset{\sim}{A}$, if $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} < \infty$.

<u>Theorem4.4</u> $f \in \underset{\sim}{M}^+$ is fuzzy integrable on $\underset{\sim}{A}$, if and only if there exists $\alpha \in [o, \infty)$, such that $\underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\alpha) < \infty$.

<u>Theorem4.5</u> Let $\underset{\sim}{A} \in \underset{\sim}{\mathcal{F}}$, $\alpha \in [o, \infty)$, then

(1) $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} \geqslant \alpha \iff \forall \beta \in [o, \alpha)$, $\underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\beta) \geqslant \alpha$;

Hence $\underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\alpha) \geqslant \alpha \Longrightarrow \int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} \geqslant \alpha$;

(2) $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} \leqslant \alpha \iff \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\bar\alpha}) \leqslant \alpha$;Therefore $\underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\alpha) \leqslant \alpha \Rightarrow \int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} \leqslant \alpha$;

(3) $\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} = \alpha \iff \forall \beta \in [o, \alpha)$, $\underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\beta) \geqslant \alpha \geqslant \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\bar\alpha})$;

Particularly, if $\underset{\sim}{\mu}(\underset{\sim}{A}) < \infty$, then

$$\int_{\underset{\sim}{A}} fd\underset{\sim}{\mu} = \alpha \iff \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\alpha) \geqslant \alpha \geqslant \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\bar\alpha}).$$

5. Convergence for Sequence of Measurable Functions

<u>Definition5.1</u> Let $\{f_n, f\} \subset \underset{\sim}{M}$, $\underset{\sim}{A} \in \underset{\sim}{\mathcal{F}}$.

Write: $\bar{D}(x)=\begin{cases} I & \text{if } f_n(x)\longrightarrow f(x) \\ \theta & \text{if } f_n(x)\not\longrightarrow f(x) \text{ (i.e. } \lim_{n\to\infty} f_n(x)\neq f(x)\text{)}\end{cases}$ for any $x\in X$.

(1) If $\underset{\sim}{A}\subset\bar{D}$, then we say $\langle f_n\rangle$ converges to f everywhere on $\underset{\sim}{A}$, and denote it by $f_n\xrightarrow{e.} f$ on $\underset{\sim}{A}$.

(2) If there exists $\underset{\sim}{E}\in\underset{\sim}{\mathfrak{F}}$ with $\underset{\sim}{\mu}(\underset{\sim}{E})=o$, such that $f_n\xrightarrow{e.} f$ on $\underset{\sim}{A}\cap\underset{\sim}{E}^c$, then we say $\langle f_n\rangle$ converges to f almost everywhere on $\underset{\sim}{A}$, and denote it by $f_n\xrightarrow{a.e.} f$ on $\underset{\sim}{A}$.

(3) If there exists $\underset{\sim}{E}\in\underset{\sim}{\mathfrak{F}}$ with $\underset{\sim}{\mu}(\underset{\sim}{A}\cap\underset{\sim}{E}^c)=\underset{\sim}{\mu}(\underset{\sim}{A})$, such that $f_n\xrightarrow{e.} f$ on $\underset{\sim}{A}\cap\underset{\sim}{E}^c$, then we say $\langle f_n\rangle$ converges to f pseudo-almost everywhere on $\underset{\sim}{A}$, and denote it by $f_n\xrightarrow{p.a.e.} f$ on $\underset{\sim}{A}$.

<u>Definition5.2</u> Let $\langle f_n, f\rangle\subset\underset{\sim}{M}$, $\underset{\sim}{A}\in\underset{\sim}{\mathfrak{F}}$.

Write: $T^n_\alpha(x)=\begin{cases} I & \text{if } |f_n(x)-f(x)|\geqslant\alpha \\ \theta & \text{if } |f_n(x)-f(x)|<\alpha\end{cases}$ for any $x\in X$,

where $\alpha\in[o,\infty)$.

(1) If for any given $\varepsilon>o$, $\underset{\sim}{\mu}(\underset{\sim}{A}\cap T^n_\varepsilon)\longrightarrow o$ as $n\longrightarrow\infty$, then we say $\langle f_n\rangle$ converges in fuzzy measure $\underset{\sim}{\mu}$ to f on $\underset{\sim}{A}$, denote it by $f_n\xrightarrow{\underset{\sim}{\mu}} f$ on $\underset{\sim}{A}$.

(2) If for any given $\varepsilon>o$, when $n\to\infty$ we have $\underset{\sim}{\mu}(\underset{\sim}{A}\cap(T^n_\varepsilon)^c)\longrightarrow\underset{\sim}{\mu}(\underset{\sim}{A})$, then we say $\langle f_n\rangle$ converges pseudo-in fuzzy measure $\underset{\sim}{\mu}$ to f on $\underset{\sim}{A}$, and denote it by $f_n\xrightarrow{p.\underset{\sim}{\mu}} f$ on $\underset{\sim}{A}$.

(3) $\langle f_n\rangle$ is said to F-mean converge to f on $\underset{\sim}{A}$, if we have

$$\lim_{n\to\infty}\int_{\underset{\sim}{A}}|f_n-f|\,d\underset{\sim}{\mu}=o.$$

<u>Proposition5.3</u> If $f_n\xrightarrow{a.e.} f$ on $\underset{\sim}{A}$, $\underset{\sim}{\mu}$ is null-subtractive, then $f_n\xrightarrow{p.a.e.} f$ on $\underset{\sim}{A}$.

<u>Proposition5.4</u> If $f_n\xrightarrow{\underset{\sim}{\mu}} f$ on X, $\underset{\sim}{\mu}$ is autocontinuous from below, then for any $\underset{\sim}{A}\in\underset{\sim}{\mathfrak{F}}$, we have $f_n\xrightarrow{p.\underset{\sim}{\mu}} f$ on $\underset{\sim}{A}$.

<u>Theorem5.5</u> F-mean convergence is equivalent to convergence in fuzzy measure.

6. Monotone Convergence Theorems for Sequence of Fuzzy Integrals

Denote:

$F^n_\alpha(x)=\begin{cases} I & \text{if } f_n(x)\geqslant\alpha \\ \theta & \text{if } f_n(x)<\alpha\end{cases}$ for any $x\in X$,

$$F_{\bar{\alpha}}^{n}(x)=\begin{cases} I & \text{if } f_n(x)>\alpha \\ \theta & \text{if } f_n(x)\leq\alpha \end{cases} \quad \text{for any } x\in X,$$

where $\alpha\in(o,\infty)$.

<u>Proposition6.1</u> Let $\langle f_n, f\rangle\subset M^+$, $A\in\mathfrak{F}$.

(1) If $f_n\searrow f$ on A, then $\bigcap_{n=1}^{\infty}(A\cap F_{\alpha}^{n})=A\cap F_{\alpha}$;

(2) If $f_n\nearrow f$ on A, then $\bigcup_{n=1}^{\infty}(A\cap F_{\bar{\alpha}}^{n})=A\cap F_{\bar{\alpha}}$.

<u>Theorem6.2</u> (Monotone Convergence Theorem) Let $\langle f_n, f\rangle\subset M^+$, $A\in\mathfrak{F}$, if $f_n\nearrow f$ on A, then $\int_A f_n d\mu\nearrow\int_A f d\mu$.

<u>Theorem6.3</u> (Monotone Convergence Theorem) Let $\langle f_n, f\rangle\subset M^+$, $A\in\mathfrak{F}$. If $f_n\searrow f$ on A, and there exist n_0 and a constant $c\leq\int_A f d\mu$ $(o\leq c)$, such that $\mu(A\cap F_{\bar{c}}^{n_0})<\infty$, then

$$\int_A f_n d\mu\searrow\int_A f d\mu.$$

<u>7. Everywhere and (pseudo-) Almost Everywhere Convergence Theorems for Sequence of Fuzzy Integrals</u>

Write: $g_n=\inf_{i\geq n} f_i$, $h_n=\sup_{i\geq n} f_i$,

$$H_{\bar{\alpha}}^{n}(x)=\begin{cases} I & \text{if } h_n(x)>\alpha \\ \theta & \text{if } h_n(x)\leq\alpha \end{cases} \quad \text{for any } x\in X,$$

where $\alpha\in[o,\infty)$.

<u>Theorem7.1</u> (Everywhere Convergence Theorem) Let $\langle f_n, f\rangle\subset M^+$, $A\in\mathfrak{F}$. If $f_n\xrightarrow{e.}f$ on A, and there exist n_0 and a constant $c\leq\int_A f d\mu$ $(o\leq c)$, such that $\mu(A\cap H_{\bar{c}}^{n_0})<\infty$, then $\lim_{n\to\infty}\int_A f_n d\mu$ is existent, and $\lim_{n\to\infty}\int_A f_n d\mu=\int_A f d\mu$.

<u>Theorem7.2</u> Let μ be null-subtractive (resp. μ be pseudo-null-subtractive with respect to A, where $A\in\mathfrak{F}$). Then for any $B\in\mathfrak{F}$, we have $\int_{A\cap B^c} f d\mu=\int_A f d\mu$ whenever $\mu(B)=o$ (resp. $\mu(A\cap B^c)=\mu(A)<\infty$).

<u>Theorem7.3</u> (Almost Everywhere Convergence Theorem) Let $A\in\mathfrak{F}$, $\langle f_n, f\rangle\subset M^+$, μ be null-subtractive. If $f_n\xrightarrow{a.e.}f$ on A, and there

exist n_o and a constant $c \leqslant \int_{A} f d\mu$ $(o \leqslant c)$, such that $\mu(A \cap H_{c}^{n_o}) < \infty$, then $\lim_{n\to\infty} \int_{A} f_n d\mu$ is existent, and $\lim_{n\to\infty} \int_{A} f_n d\mu = \int_{A} f d\mu$.

Theorem7.4 (Pseudo-Almost Everywhere Convergence Theorem) Let $\{f_n, f\} \subset M^+$, $A \in \mathfrak{F}$, $\mu(A) < \infty$, μ be pseudo-null-subtractive with respect to A. If $f_n \xrightarrow{p.a.e.} f$ on A, then $\lim_{n\to\infty} \int_{A} f_n d\mu$ is existent, and $\lim_{n\to\infty} \int_{A} f_n d\mu = \int_{A} f d\mu$.

8. Convergence (Pseudo-) in Fuzzy Measure Theorem for Sequence of Fuzzy Integrals

Proposition8.1 Let $\{f_n, f\} \subset M^+$, $b, c \in (o, \infty)$, then

(1) $F_{c+2b}^{n} \subset F_{c+b} \cup T_{b}^{n}$;

(2) $F_{c-b}^{n} \supset F_{c} \cap (T_{b}^{n})^{c}$ (where $b < c$).

Theorem8.2 (Convergence Pseudo-in Fuzzy Measure Theorem) Let $\{f_n, f\} \subset M^+$, $A \in \mathfrak{F}$, $\mu(A) < \infty$, μ be pseudo-autocontinuous with respect to A. If $f_n \xrightarrow{p.\mu} f$ on A, then $\lim_{n\to\infty} \int_{A} f_n d\mu$ is existent, and $\lim_{n\to\infty} \int_{A} f_n d\mu = \int_{A} f d\mu$.

Theorem8.3 (Convergence in Fuzzy Measure Theorem) Let $\{f_n, f\} \subset M^+$, μ be autocontinuous. If $f_n \xrightarrow{\mu} f$ on X, then for any $A \in \mathfrak{F}$, we have $\lim_{n\to\infty} \int_{A} f_n d\mu = \int_{A} f d\mu$.

Corollary8.4 (F-Mean Convergence Theorem) Let $\{f_n, f\} \subset M^+$, μ be autocontinuous. If $\{f_n\}$ F-mean converges to f on X, then for any $A \in \mathfrak{F}$, we have $\lim_{n\to\infty} \int_{A} f_n d\mu = \int_{A} f d\mu$.

Remark: If we take $L = [o, 1]$, the conclusions given in this paper are identical with those proved in [6,7,12].

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