EXTENSION OF POSSIBILITY MEASURE DEFINED ON AN ARBITRARY NON-EMPTY CLASS OF L-FUZZY SETS

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Abstract

The concepts of possibility measure and P-consistency of set function defined on L-fuzzy sets are introduced and necessary and sufficient condition of the extension of possibility measure on L-fuzzy sets.

Keywords: L-fuzzy set, Possibility measure,
P-consistency of set function.

In this paper, let X be a set, $F_L(X) = \{ \underline{A};\ \underline{A}: X \longrightarrow L$, L is a complete lattice $\}$, and C* be an arbitrary nonempty subset of $F_L(X)$, u be a mapping from C* into the unit interval [0 , 1], and we make the following convention: $\bigcup_{\emptyset}\{\cdot\} = \emptyset$, $\sup_{\emptyset}\{\mu(\cdot)\} = 0$, $\inf_{\emptyset}\{\mu(\cdot)\} = 1$.

Definition 1. A possibility measure on $F_L(X)$ is a non-negative real valued set function $\pi : F_L(X) \longrightarrow [0 , 1]$ with the property:

$$\pi(\bigcup_{t\in T}\underline{A}_t) = \sup_{t\in T}\pi(\underline{A}_t),$$ whenever $\{\underline{A}_t;\ t\in T\}\subset F_L(X)$,

where T is an arbitrary index set.

Definition 2. $\mu: C^* \longrightarrow [0\ ,\ 1]$ is called P-consistent, if for every $\{\underline{A}_t;\ t\in T\}\subset C^*$, $\underline{A}\in C^*$, with $\underline{A}\subset\bigcup_{t\in T}\underline{A}_t$, we have

$$\mu(\underline{A}) = \sup_{t\in T}\mu(\underline{A}_t),$$

where T is an arbitrary index set.

Theorem 1. μ can be extended to a possibility measure on $F_L(X)$, if and only if μ is P-consistent.

Proof. Necessity. Obvious.

Sufficiency. If we define

$$\pi : F_L(X) \longrightarrow [0\ ,\ 1]$$

$$\underline{B} \longmapsto \sup_{x\in X}\ \inf_{\substack{(\bigcup_{s\in S^x}\underline{E}_s)(x)\geq\underline{B}(x)\\ \underline{E}_s\in C^*}}\ \sup_{s\in S^x}\mu(\underline{E}_s), \tag{1}$$

where S^x is an arbitrary index set, then π is a possibility measure on $F_L(X)$, and a extension of μ on C^*. To conclude the assertions, we first prove that π is a possibility measure. In fact, the monotonicity of π is obvious. By the monotonicity of π , we have, for every $\{\underline{A}_t;\ t\in T\}\subset F_L(X)$,

$$\pi(\underline{A}_t) \leq \pi(\bigcup_{t\in T}\underline{A}_t),$$

and hence

$$\sup_{t\in T}\pi(\underline{A}_t) \leq \pi(\bigcup_{t\in T}\underline{A}_t),$$

where T is an arbitrary index set.

On the other hand, 1) when for every $x\in X$, $t\in T$ there exists $\{\underline{E}_s;\ s \in S^x\} \subset C^*$ such that

$$(\bigcup_{s\in S^x}\underline{E}_s)(x) \geq \underline{A}_t(x)$$

and for any $\varepsilon > 0$

$$\inf_{\substack{(\bigcup_{s\in S_t^x}\underline{E}_s)(x) \geq \underline{A}_t(x) \\ \underline{E}_s \in C^*}} \sup_{s\in S^{x'}} \mu(\underline{E}_s) = \sup_{s\in S^{x'}} \mu(\underline{E}_s) - \varepsilon ,$$

then, since

$$(\bigcup_{t\in T}\bigcup_{s\in S_t^x}\underline{E}_s)(x) \geq \bigvee_{t\in T}\underline{A}_t(x) = (\bigcup_{t\in T}\underline{A}_t)(x),$$

we have

$$\sup_{t\in T} \inf_{\substack{(\bigcup_{s\in S_t^x}\underline{E}_s)(x)\geq\underline{A}_t(x) \\ \underline{E}_s \in C^*}} \sup_{s\in S_t^{x'}} \mu(\underline{E}_s) = \sup_{t\in T} \sup_{s\in S_t^{x'}} \mu(\underline{E}_s) - \varepsilon$$

$$= \sup_{\substack{s\in \bigcup S_t^{x'} \\ t\in T}} \mu(\underline{E}_s) - \varepsilon = \inf_{\substack{(\bigcup_{s\in S^x}\underline{E}_s)(x)\geq(\bigcup_{t\in T}\underline{A}_t)(x) \\ \underline{E}_s \in C^*}} \sup_{s\in S^{x'}} \mu(\underline{E}_s) - \varepsilon$$

this shows that

$$\sup_{x\in X}\sup_{t\in T} \inf_{\substack{(\bigcup_{s\in S_t^x}\underline{E}_s)(x)\geq\underline{A}_t(x) \\ \underline{E}_s \in C^*}} \sup_{s\in S_t^{x'}} \mu(\underline{E}_s) \qquad (*)$$

$$= \sup_{x\in X} \inf_{\substack{(\bigcup_{s\in S^x}\underline{E}_s)(x)\geq(\bigcup_{t\in T}\underline{A}_t)(x) \\ \underline{E}_s \in C^*}} \sup_{s\in S^{x'}} \mu(\underline{E}_s) - \varepsilon$$

2) When there exists $x_o \in X$, $t_o \in T$, for every $\{\underline{E}_s;\ s \in S_{t_o}^{x_o}\}$ $\subset C^*$ such that

$$(\bigcup_{s\in S_{t_o}^{x_o}}\underline{E}_s)(x_o) \not\geq \underline{A}_{t_o}(x_o),$$

by using $\inf_{\emptyset}\{\mu(\cdot)\} = 1$, $\sup_{\emptyset}\{\mu(\cdot)\} = 0$, (*) is also true.

~~It~~ yields that

$$\sup_{t\in T} \pi(\underline{A}_t) \geq \pi(\bigcup_{t\in T} \underline{A}_t).$$

Consequently,

$$\sup_{t\in T} \pi(\underline{A}_t) = \pi(\bigcup_{t\in T} \underline{A}_t),$$

which means π is a possibility measure.

Next, we prove that π is an extension of μ on C*. In fact, for every $\underline{B} \in C^*$, we have

$$\pi(\underline{B}) = \sup_{x\in X} \inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x)\geq \underline{B}(x) \\ \underline{E}_s \in C^*}} \sup_{s\in S^{x'}} u(\underline{E}_s) \leq \sup_{x\in X'} u(\underline{B}) = \mu(\underline{B}).$$

On the other hand, for any $\varepsilon > 0$, every $x \in X$ there exists $\{\underline{E}_s;\ s \in S^x\} \subset C^*$ such that

$$\underline{B}(x) \leq (\bigcup_{s\in S^x} \underline{E}_s)(x) \leq (\bigcup_{s \in \bigcup_{x\in X} S^x} \underline{E}_s)(x),$$

and

$$\inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x)\geq \underline{B}(x) \\ \underline{E}_s \in C^*}} \sup_{s\in S^{x'}} u(\underline{E}_s) \geq \sup_{s\in S^x} u(\underline{E}_s) - \varepsilon,$$

hence, by using the P-consistence of u,

$$\pi(\underline{B}) = \sup_{x\in X} \inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x)\geq \underline{B}(x) \\ \underline{E}_s \in C^*}} \sup_{s\in S^{x'}} u(\underline{E}_s)$$

$$\geq \sup_{x\in X} \sup_{s\in S^x} \mu(\underline{E}_s) - \varepsilon = \sup_{s \in \bigcup_{x\in X} S^x} \mu(\underline{E}_s) - \varepsilon \geq \mu(\underline{B}) - \varepsilon,$$

therefore

$$\pi(\underline{B}) \geq \mu(\underline{B}).$$

Consequently,

$$\pi(\underline{B}) = \mu(\underline{B}),$$

and we complete the proof of the theorem.

In usual case, the extension of a mapping μ with P-consistent from an arbitrary nonempty class of the L-fuzzy subsets of X into the unit interval $[0, 1]$ to a possibility measure on $F_L(X)$ may not be unique. All extensions of possibility measure is denoted $E\pi(\mu)$. By using theorem 1, we know that $E\pi(\mu)$ is nonempty, if μ is P-consistent.

For two mappings $\mu_1: F_L(X) \longrightarrow [0, 1]$ and $\mu_2: F_L(X) \longrightarrow [0, 1]$, we define ordering relation " $\leq$ ":

$$\mu_1 \leq \mu_2 \text{ if and only if } \mu_1(\underline{A}) \leq \mu_2(\underline{A}), \text{ for every } \underline{A} \in F_L(X).$$

It is easy to prove that " $\leq$ " is a partial ordering relation on $E\pi(\mu)$. Therefore the least upper bound of $\mu_1, \mu_2 \in E\pi(\mu)$ can be defined by

$$(\sup\{\mu_1, \mu_2\})(\underline{A}) = \mu_1(\underline{A}) \vee \mu_2(\underline{A}), \text{ for all } \underline{A} \in F_L(X).$$

<u>Theorem 2.</u> $(E\pi(\mu), \leq)$ is an upper semi-lattice, and the extension π defined by (1) is the greatest element of $E\pi(\mu)$.

Proof. 1) Obviously, $(E\pi(\mu), \leq)$ is an upper semi-lattice.

2) The extension π defined by (1) is the greatest element of the $E\pi(\mu)$. For arbitrary $\pi' \in E\pi(\mu)$, $\underline{B} \in F_L(X)$, we define

$$\underline{F}_x(y) = \begin{cases} \inf\limits_{\substack{(\bigcup_{s\in S^x}\underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C^*}} (\bigcup_{s\in S^x}\underline{E}_s)(x) & \text{if } y = x; \\ 0 & \text{if } y \neq x, \end{cases}$$

for every $x \in X$. If $\underline{B}(x) \leq (\bigcup_{s\in S^x}\underline{E}_s)(x)$, $\underline{E}_s \in C^*$, we have

$$\underline{F}_x \subset \bigcup_{s \in S^x} \underline{E}_s,$$

hence

$$\pi'(\underline{F}_x) \leq \sup_{s \in S^x} \pi'(\underline{E}_s),$$

therefore

$$\pi'(\underline{F}_x) \leq \inf_{\substack{(\bigcup_{s \in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C^*}} \sup_{s \in S^x} \pi'(\underline{E}_s)$$

$$= \inf_{\substack{(\bigcup_{s \in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C^*}} \sup_{s \in S^x} \mu(\underline{E}_s),$$

for every $x \in X$, it follows, by using

$$\underline{B}(x) \leq \inf_{\substack{(\bigcup_{s \in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C^*}} (\bigcup_{s \in S^x} \underline{E}_s)(x) = \underline{F}_x(x) \leq (\bigcup_{x \in X} \underline{F}_x)(x),$$

that

$$\pi(\underline{B}) = \sup_{x \in X} \inf_{\substack{(\bigcup_{s \in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \quad C^*}} \sup_{s \in S^x} \mu(\underline{E}_s) \geq \sup_{x \in X} \pi'(\underline{F}_x)$$

$$= \pi'(\bigcup_{x \in X} \underline{F}_x) \geq \pi'(\underline{B}).$$

References

[1] Zhang Guangquan, Semi-lattice structure of all extensions of possibility measure and consonant belief function on the fuzzy set, BUSEFAL, 33(1987), 95-103.

[2] Zhang Guangquan, The extensions of a class of Sc-fuzzy measure, Hebei University, 3(1987), 11-17.

[3] Qiao Zhong, On the extension of the possibility measures, Hebei University, 3(1987), 66-70.

[4] Wang Zhenyuan, Extension of possibility measures defined on an arbitrary nonempty class of sets,(Presented at the First I,F,S.A. congress, Palma de Mallorca, Spain, 1985).

[5] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1(1978), 3-28.

[6] D. Dubois and H. Prade, Fuzzy Sets and Systems: Theory and Applications(Academic Press, New York, 1980).

[7] Wang Zhenyuan, Semi-lattice isomorphism of the extensions of possibility measure and the solutions of fuzzy relation equation, in Cybernetics and Systems '86 (R. Trappl ed.), D. Reidel Publishing Company, (1986) 581-583.