Completeness in residuated lattices

Alexej Kolcun

Czechoslovak Academy of Sciences, Mining Institute
Studentská 1768, 708 00 Ostrava-Poruba, Czechoslovakia

Introduction

Let a residuated lattice L = $\langle L, \vee, \wedge, \longrightarrow, \otimes, 1, 0\rangle$ be given, i.e.:

a) $\langle L; \wedge, \vee, 0, 1\rangle$ is a complete distributive lattice,

b) $\langle L; \otimes, 1\rangle$ is a comutative monoid,

c) $\otimes$ is an isotone binary operation on L in both variables,

d) $\longrightarrow$ is a binary operation on the set L which is antitone in the first and isotone in the second variable respectively,

e) $\langle \otimes, \longrightarrow\rangle$ is an adjoint pair of the operations, i.e.

$$(\forall\ a,b,c \in L):\ (a\otimes b \le c \iff a \le b\rightarrow c).$$

We define the operation of the biresiduum as follows:

$$a \leftrightarrow b \equiv (a \longrightarrow b) \wedge (b \longrightarrow a).$$

We say that the function $f: L^n \longrightarrow L$ fits the residuated lattice L, iff the following holds:

$$(\exists\ k \in \mathbb{N}^n)\ (\forall\ x,y \in L^n):\ \bigotimes_{i=1}^{n} (x_i \leftrightarrow y_i)^{k_i} \le f(x) \leftrightarrow f(y)$$

where $a^k \equiv a \otimes \ldots \otimes a$ – k-times, $\mathbf{x} = (x_1 \ldots x_n)$, $\mathbf{y} = (y_1 \ldots y_n)$ and $\mathbf{k} = (k_1 \ldots k_n)$.

If the function $f: L^n \longrightarrow L$ fits the lattice L then we write

$$f \mathrel{\hat{\in}} L\ .$$

Let O be a set of functions fitting L. Then $\langle L; \wedge, \vee, \otimes, \longrightarrow, 0, 1, O\rangle$ is an enriched residuated lattice.

In this paper we deal with the problem of completenes of residuated lattices. We answer the following questions :

1. Is it possible to express all the fitting functions using only the basic operations of the residuated lattice ?

2. If the answer to 1. is negative, what functions must be added to the residuated lattice to be able to express all the fitting functions ?

If we consider continous functions on the bicompact spaces then Weierstrasse-Stone theorem or its Kakudaki-Kreyn corollary are solutions of the completenes problem (see [5]).

In this theorem, however, the ring of the functions with standard operations of addition and multiplication are considered. But no finite subset $B \subset (0,1)$ with the operation of the multiplication forms a subalgerba of the algebra $\langle\langle 0,1\rangle; \,.\, \rangle$. Therefore, the canonical form of the functions according the Weierstrasse theorem cannot be used if the domain of the considered functions is a discrete set.

In this paper, we start from the canonical form of discrete functions. This form will then be generalized to the case of the continous ones.

1. Basic definitions and relations

We introduce the following symbols:

Φ is a set of operations on the set L, $\mathbf{A}$ is an algebra $\mathbf{A} = \langle L; \Phi\rangle$. By $P \in [\mathbf{A}]$ we denote that $P\colon L^n \longrightarrow L$ is created by superposition of operations from $\mathbf{A}$, and

$$f \leftrightarrow P \equiv \bigwedge_{x \in L^n} (f(x) \leftrightarrow P(x))$$

where $f\colon L^n \longrightarrow L$.

1.1. Definition . The algebra $\mathbf{A} = \langle L; \Phi \rangle$ is functionaly complete in the residuated lattice $\mathbf{L} = \langle L; \wedge, \vee, \otimes, \longrightarrow, 0, 1\rangle$, if

$$(\forall f \,\hat{\in}\, \mathbf{L})(\forall \eta < 1)(\exists P \in [\mathbf{A}])(f \leftrightarrow P > \eta)$$

holds.

1.2. Theorem . Let $\mathbf{L} = \langle L; \wedge, \vee, 0, 1\rangle$ be a complete lattice. Then every function $f\colon L^n \longrightarrow L$ can be expressed in the form of a disjunctive normal form (DNF):

$$f(x) = \bigvee_{a \in L^n} (f(a) \wedge \bigwedge_{i=1}^{n} J_{a_i}(x_i)) \,, \qquad (1)$$

where

$$J_a(x) = \begin{cases} 0 & \text{if } x \neq a \\ 1 & \text{if } x = a, \end{cases}$$

i.e. the function $f(x)$ can be described on the basis of the system of the functions $\{J_a(x) : a \in L\}$ and constants $\{f(a) : a \in L^n\}$

P r o o f . See [1]. ■

1.3. Theorem . Let L be a residuated lattice and Φ be a set of functions fitting the lattice L. Then

$$f \in [\langle L;\Phi\rangle] \Rightarrow f \,\hat{\in}\, L \;,$$

(i.e. the superposition of fitting functions in L, fits the lattice L).

P r o o f . See [3]. ■

1.4. Definition . L is a continous residuated lattice if the operation of residuum $\rightarrow$ is continous.

1.5. Definition . The residuated lattice L with $L = \langle 0,1\rangle$ and the operations of multiplication and residuation defined by $a \otimes b = 0 \vee (a+b-1)$, $a \rightarrow b = 1 \wedge (1-a+b)$, is called Łukasiewicz interval. We denote it by Ł.

1.6. Remark . The operation of biresiduum in Ł can be expressed by

$$a \leftrightarrow b = 1 \wedge (1-a+b) \wedge (1-b+a) = 1 \wedge (1-|a-b|) \;.$$

1.7. Theorem . Every continous residuated lattice $L = \langle\langle 0,1\rangle;\ \wedge,\ \vee,\ \otimes',\ \rightarrow',\ 0\ ,1\rangle$ is isomorphic with the Łukasiewicz interval $Ł = \langle\ \langle 0,1\rangle;\ \wedge,\ \vee,\ \otimes,\ \rightarrow,\ 0,\ 1\ \rangle$, i.e. there is an isotone one-to-one function ϕ: $\langle 0,\ 1\rangle \longrightarrow \langle 0,\ 1\rangle$ such that

$$\phi(x \otimes' y) = \phi(x) \otimes \phi(y)$$

holds for every $x, y \in \langle 0,1\rangle$.

P r o o f . See [4]. ■

1.8. Theorem . Let $f \,\hat{\in}\, Ł$. Then f is a continous function.

P r o o f . See [3]. ■

1.9. Definition . Let $L_k = \{0 = l_0 < l_1 < \ldots < l_k = 1\}$, and put $l_i \otimes l_j = l_{0 \vee (i+j-k)}$, $l_i \rightarrow l_j = l_{k \wedge (k-i+j)}$.
Then the residuated lattice $Ł_k = \langle L;\ \wedge,\ \vee,\ \otimes,\ \rightarrow,\ 0,\ 1\rangle$ is

called the Łukasiewicz chain.

1.10. Theorem. Every function $f\colon L_k^r \longrightarrow L_k$ fits the lattice $Ł_k$.

P r o o f . See [3]. ∎

2. Functional incompletenes of the algebra L in the residuated lattice L

In this section, we demonstrate that no residuated lattice L is complete in itself, i.e. the fitting functions in L cannot be expressed only on the basis of the operations in L.

2.1. Definition . A function $f\colon A^n \longrightarrow A$ preserves a set $B \subseteq A$, if

$$\{f(x) : x \in B^n\} \subseteq B.$$

A set Φ of functions preserves a set B if every function $f \in \Phi$ preserves a set B.

2.2. Lemma . Let L be a residuated lattice. Then

$$a \longrightarrow 0 \equiv \neg a < 1 \quad \text{if} \quad a > 0$$
$$a \longrightarrow 1 = a$$
$$0 \longrightarrow a = 1$$
$$1 \longrightarrow a = a$$
$$a \longrightarrow a = 1$$

holds for all $a \in L$

P r o o f . The following relations hold for every $a, b \in L$ (see [2]):

$$a \otimes b \le a, \; a \otimes b \le b \tag{2}$$

$$a \longrightarrow b = \vee\{\omega \in L : a \otimes \omega \le b\} \tag{3}$$

If $a > 0$, then $a \to 0 = \vee\{\omega : a \otimes \omega \le 0\} < 1$ since $a \otimes 1 = a$.
$a \longrightarrow 1 = \vee\{\omega\colon a\otimes\omega \le 1\} = 1$ since L is a complete lattice.
$0 \longrightarrow a = \vee\{\omega : 0\otimes\omega \le a\} = 1$ because $0\otimes\omega = 0$ for every ω.
$1 \longrightarrow a = \vee\{\omega : 1\otimes\omega \le a\} = a$ because $1\otimes\omega = \omega$.
$a \longrightarrow a = \vee\{\omega : a\otimes\omega \le a\} = 1$ because $a\otimes 1 = a$. ∎

2.3. Lemma . The set [L] preserves the set $\{0,1\}$.

P r o o f . The operations $\wedge$, $\vee$, 0, 1 obviously preserve the set $\{0,1\}$.

Since $\langle L;\otimes,1\rangle$ is monoid, it follows from (2) that $\otimes$

preserves the set $\{0,1\}$.

If $a \in \{0, 1\}$ then it follows from lemma 2.2 that $\longrightarrow$ preserves the set $\{0,1\}$.

A superposition of functions preserving the set A obviously preserves the set A, which implies that the assertion of lemma is true. ■

2.4. Theorem . The algebra **L** is not functionaly complete in residuated lattice **L**.

P r o o f . Since $c \longrightarrow c = 1$ holds for every $c \in L$ (see 2.2), every constant function $f(x) = c \in L$ fits the lattice **L**.

We will demonstrate that no constant different from **0**, **1** can be approximated by functions from **L**.

Let $P \in [L]$ and $c < 1$. Then

$$P \leftrightarrow c \equiv \bigwedge_{\mathbf{x} \in L^n} (P(\mathbf{x}) \leftrightarrow c) \leq P(\mathbf{y}) \leftrightarrow c$$

holds for every $\mathbf{y} \in L^n$.

Set $\mathbf{y}=\mathbf{0}$. Then we obtain

$$P \leftrightarrow c \leq P(\mathbf{0}) \leftrightarrow c .$$

It follows from lemma 2.3 that $P(\mathbf{0}) \in \{0,1\}$. The inequality

$$P(\mathbf{0}) \leftrightarrow c = (P(\mathbf{0}) \rightarrow c) \wedge (c \rightarrow P(\mathbf{0})) \leq c \wedge \neg c$$

then follows from lemma 2.2 which results in

$$P(\mathbf{0}) \leftrightarrow c \leq c \neg c < 1 \qquad (3)$$

for $c \in L \setminus \{0, 1\}$.

Since (3) holds for every function $P \in [L]$, the theorem is proved. ■

3. Functionaly complete algebras

In this section, we construct a functionaly complete algebra for the residuated lattice $Ł_k$. This result is then generalised to the Ł. We will use the following notation :

$$\neg x \equiv x \longrightarrow 0 = 1 \wedge (1 - x) = 1 - x$$

$$x \oplus y \equiv (\neg x) \longrightarrow y = 1 \wedge (x + y)$$

$$nx \equiv x \oplus \ldots \oplus x = 1 \wedge (n.x)$$

$$x^n \equiv x \otimes \ldots \otimes x = 0 \vee (n.x - (n - 1))$$

In $Ł_k$ we obtain

$$1_i^n \equiv 1_i \otimes \ldots \otimes 1_i = 1_{0 \vee (ni-k(n-1))}$$

3.1. Theorem . The algebra

$$\mathbf{A}_k = \langle L_k;\ \wedge,\ \vee,\ \otimes,\ \longrightarrow,\ \{a : a \in L_k\}\rangle$$

is functionaly complete in residuated lattice $Ł_k$.

P r o o f . It follows from theorem 1.10 that every function f: $L_k^n \longrightarrow L_k$ fits $Ł_k$. We will demonstrate that each f can be described using some superposition of fitting functions.

According to theorem 1.2, every function in a complete lattice (and, thus in $Ł_k$ as well) can be expressed using a DNF (1). The vector $\mathbf{a} = (a_1, a_2, \ldots a_n)$ in (1) does not depend on the variables $\mathbf{x} = (x_1 \ldots x_n)$, and so f(**a**) is a constant function. Therefore f(**a**) fits $Ł_k$. We will demonstrate that it is possible to express the function $J_{\mathbf{a}}(x)$ using a superposition of constants and functions from $Ł_k$ and so $J_{\mathbf{a}}(x)$ fits $Ł_k$ according to theorem 1.3.
It follows from the definition of the residuum in $Ł_k$ that:

$$a \leftrightarrow x = 1_k = 1 \quad \text{if } x=a$$

$$a \leftrightarrow x \leq 1_{k-1} \quad \text{if } x \neq a$$

$$1_{k-1}^n = 1_{0 \vee (n(k-1)-k(n-1))}$$

For $n \geq k$ we obtain

$$1_{k-1}^n = 1_0 = 0 \ .$$

Then

$$J_{\mathbf{a}}(x) = (x \leftrightarrow a)^k \ ,$$

holds for every a, $x \in L_k$. Using theorem 1.3 we verify that the assertion of the theorem is true. ∎

3.2. Theorem. The algebra

$$\mathbf{A} = \langle\langle 0,1\rangle;\ \wedge,\ \vee,\ \otimes,\ \longrightarrow,\ \{a : a \in \langle 0,1\rangle\}\rangle$$

is functionaly complete in Ł.

P r o o f . According to theorem 1.8 it is sufficient to prove that every continous function can be aproximated with an arbitrary precision by some superposition of functions from **A**.

Let

$$\phi_a(k,x) = 2(x \leftrightarrow a)^k \qquad (4)$$

where a, $x \in L$ and $k \in \mathbb{N}$.

$\phi_a(k,x)$ is analogous to the function $J_a(x)$ from the previous theorem. It can be seen from (4) that it is a superposition of the constants and of the operations on Ł and so it fits Ł.

Consider a set of equidistant points

$$B_k = \{a_i : a_i = i a_1\} \qquad (5)$$

where $k \in \mathbb{N}$, i = 0, 1, 2,...k and $a = \vee \{x \in \langle 0,1\rangle : kx \le 1\}$

We can see from (5) that every set B_k can be constructed on the basis of constants and operations in L.

Let

$$F_n(x) = \bigvee_{i=1}^{n} (\phi_{a_i}(n,x) \wedge f(a_i)) \qquad (6)$$

where $a_i \in B_n$.

$F_n(x)$ obviously fits Ł.

In Ł the relations (4), (5) have the next form :

$$\phi_a(k,x) = \begin{cases} 1 & |x - a| \le \frac{1}{2k} \\ 2 - 2k|x - a| & \frac{1}{2k} \le |x - a| \le \frac{1}{k} \\ 0 & |x - a| \ge \frac{1}{k} \end{cases} \qquad (4')$$

$$B_k = \{ 0, \frac{1}{k}, \frac{2}{k}, \ldots \frac{k-1}{k}, 1 \} \qquad (5')$$

It can be seen from (4') and (6) that, in the interval $\langle a_i, a_{i+1}\rangle$, $F_n(x)$ depends only on the functions $\phi_{a_i}(n,x)$, $\phi_{a_{i+1}}(n, x)$. Assume $f(a_i) \le f(a_{i+1})$ (the case $f(a_{i+1}) \ge f(a_i)$ is examined analogously). In Ł, we have

$$F_n(x) = \begin{cases} f(a_i) & x\in I_1 \\ \phi_{a_{i+1}}(n,x) & x\in I_2 \\ f(a_{i+1}) & x\in I_3 \end{cases} = \begin{cases} f(\frac{i}{n}) & x\in I_1 \\ 2nx - 2i & x\in I_2 \\ f(\frac{i+1}{n}) & x\in I_3 \end{cases} \qquad (6')$$

where

$$I_1 = \langle a_i,\ \bigwedge\{x: \phi_{a_{i+1}}(n,x) \geq \phi_{a_i}(n,x)\}\rangle = \langle \tfrac{i}{n},\ \tfrac{i}{n} + \tfrac{1}{2n} f(\tfrac{i}{n}) \rangle$$

$$I_2 = \langle \bigwedge\{x: \phi_{a_{i+1}}(n,x) \geq \phi_{a_i}(n,x)\},\ \bigwedge\{x: \phi_{a_{i+1}}(n,x) \geq f(a_{i+1})\} \rangle =$$

$$= \langle \tfrac{i}{n} + \tfrac{1}{2n} f(\tfrac{i}{n}),\ \tfrac{i}{n} + \tfrac{1}{2n} f(\tfrac{i+1}{n}) \rangle$$

$$I_3 = \langle \bigwedge\{x: \phi_{a_{i+1}}(n,x) \geq f(a_{i+1})\},\ a_{i+1}\rangle = \langle \tfrac{i}{n} + \tfrac{1}{2n} f(\tfrac{i+1}{n}),\ \tfrac{i+1}{n} \rangle$$

It will be demonstrated that

$$(\forall\ \eta<1)\ (\exists\ n\in \mathbb{N})\ :\ \bigwedge_{x\in\langle 0,1\rangle} (F_n(x) \leftrightarrow f(x)) > \eta\ . \qquad (7)$$

It follows from remark 1.6 that (7) is equivalent to the condition

$$(\forall \varepsilon>0)(\exists n \in \mathbb{N})\ :\ (\,|f(x) - F_n(x)|\, < \varepsilon) \qquad (8)$$

The middle member of the expression (6') can be arranged as follows

$$F_n(x) = (1-\tau)f(\tfrac{i}{n}) + \tau f(\tfrac{i+1}{n}),$$

$$\text{where } \tau = 2n\ \frac{x - (\frac{i}{n} + \frac{1}{2n} f(\frac{i}{n}))}{f(\frac{i+1}{n}) - f(\frac{i}{n})} \in \langle 0,\ 1\rangle,\quad x \in I_2.$$

Let us verify whether the condition (8) is fulfilled.
Then

$$|f(x)-F_n(x)| = \begin{cases} |f(x)-f(\frac{i}{n})| & x\in I_1 \\ (1-\tau)\,|f(x)-f(\frac{i}{n})| + \tau\,|f(x)-f(\frac{i+1}{n})| & x\in I_2 \\ |f(x)-f(\frac{i+1}{n})| & x\in I_3 \end{cases} \qquad (9)$$

The function f is continous on ⟨0,1⟩ and so

$$(\forall\ \varepsilon>0),(\exists\ \delta>0)\ :\ |x - \tfrac{i}{n}| < \delta \Rightarrow |f(x) - f(\tfrac{i}{n})| < \varepsilon \qquad (i=1..n)$$

holds in the points $a \in B_n$.

If $x \in I_1 \cup I_2 \cup I_3$ then

$$\max(|x - \frac{i}{n}|, |x - \frac{i+1}{n}|) < \frac{1}{n} = \delta,$$

implies

$$\max(|f(x) - f(\frac{i}{n})|, |f(x) - f(\frac{i+1}{n})|) < \varepsilon .$$

According to (9), the conditions (8) and therefore (7) are fulfilled.

It has been demonstrated that every continous function $f: \langle 0, 1\rangle \longrightarrow \langle 0, 1\rangle$ and, due to theorem 1.8, every fitting function, can be aproximated by functions from **A** with an arbitrary precision.

The function $f: \langle 0,1\rangle^r \longrightarrow \langle 0,1\rangle$ can be examined analogously. Its aproximation is

$$F_n(x) = \bigvee_{\mathbf{a} \in B_n^r} (f(\mathbf{a}) \wedge \bigwedge_{i=1}^{r} \phi_{a_i}(n, x_i)) \qquad (10)$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_r)$, $a_i \in B_n$. ■

3.3. Conclusion . The operations $\otimes$, $\longrightarrow$ in the residuated lattice $Ł_k$ with support $L_k = \langle 0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\rangle$ are the projections of $\otimes$, $\longrightarrow$ in lattice Ł to the set L_k, and so, the algebra $\mathbf{A}_k$ from theorem 3.1 is subalgebra of algebra **A** from theorem 3.2.

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