Representations of t-norms *)

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Summary. Many ways lead to the representation of associative binary operations on a real interval. In particular we get representations of t-norms and t-conorms. Some simple consequences of these representations are presented.

1. Introduction. Let I denote a proper interval. We assume that

(1) $\quad * : I \times I \to I$ is a continuous associative binary operation.

Such operations and connected algebraic structures $(I,*)$ are subject of many investigations. We sketch here the main four ways of these investigations.

1.1. Functional equation of associativity. Treating our binary operation as a real function of two variables we can write

(2) $\quad F[F(x,y),z] = F[x,F(y,z)]$ for $x, y, z \in I$.

Equation (2) was introduced by Abel [1] and further considered by many authors (cf. [2], [3]). These investigations are summarized by Aczél [3] and [4]. Some new results were presented by Ling [17], Palés and Craigen [20], and Aczél [5].

1.2. Fully ordered semigroups. Ordered semigroups on subsets of real line were considered by Hölder [15], Cartan [8], Clifford [10] and Fuchs [12]. Results of these investigations are summarized by Fuchs [12], where three different classes of such semigroups are characterized.

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1.3. Triangular norms and conorms. Consideration of statistical metric spaces by Menger [19] brought definitions of new associative binary operations on unit interval. These operations called t-norms and t-conorms were examined by Schweizer and Sklar [21], [22], and Frank [11]. Many other papers are summarized by Schweizer and Sklar [23].

1.4. Continuous logical connectives. Continuously valued logic needs extended logical operations. Some of them were suggested by Łukasiewicz [18]. Recent consideration of this problem was initiated by Bellman and Giertz [7] in connection with calculus of fuzzy sets. Problem was taken on by Hamacher [13], Alsina, Trillas and Valverde [6], Klement [16], and Weber [24]. Many of considered problems are related to earlier investigations of associativity mentioned above.

2. Continuous real semigroups. Let $a = \inf I < b = \sup I$. We summarize properties of continuous semigroup $(I,*)$ under some additional assumptions.

2.1. Theorem (Aczél [5]). The semigroup $(I.*)$ under assumption (1) is a group iff it is order isomorphic to $(R,+)$, i.e. iff there exists a continuous increasing bijection $f : I \to R$ such that

(3) $$x * y = f^{-1}(f(x) + f(y)) \quad \text{for} \quad x, y \in I.$$

2.2. Theorem (Aczél [5]). The semigroup $(I.*)$ under assumption (1) is cancellative iff there exists a continuous injection $f : I \to R$ with property (3). Such injection f is determined uniquely up to a positive multiplicative constant. In particular, any continuous cancellative semigroup $([e,b),*)$ with unit e is order isomorphic to $([0,\infty),+)$.

A characterization of not cancellative semigroups needs some additional assumptions. Operation $*$ is *isotone* if

(4) $$x \geq y \Rightarrow (x * t \geq y * t,\ t * x \geq t * y) \quad \text{for} \quad t, x, y \in I.$$

Continuous operation $*$ is *Archimedean*(cf.Schweizer and Sklar [23]) if

(5) $$x * x \neq x \quad \text{for} \quad x \in \text{Int } I,\ x \neq e.$$

Operation $*$ is *idempotent* if

(6) $\quad x * x = x \quad \text{for } x \in I.$

2.3. Lemma (Clifford [10]). Let operation $*$ fulfil assumptions (1), (4) and (5). If the semigroup $(I,*)$ is not cancellative, then:
a) it contains a null z,
b) for every $x \neq e$ there exists a $k \in N$, such that $x^k = z$,
c) for every $u, x, y \in I$, $u * x = u * y \neq z$ implies that $x = y$.

Because of property b) the above semigroups are called nilpotent.

2.4. Theorem (cf. Fuchs [12]). The semigroup $([e.b],*)$ with unit e and properties (1), (4) is not cancellative and Archimedean iff it is order isomorphic to the semigroup $([0,1],\oplus)$, where

(7) $\quad x \oplus y = \min(1, x + y) \quad \text{for } x, y \in [0,1].$

Moreover (cf. Ling [17]), such operation $*$ has a representation (3) with formal extension of the inverse of f by the formula

$$f^{-1}(x) = b \quad \text{for } x > f(b).$$

2.5. Theorem (Czogała, Drewniak [9]). The semigroup $([a,b],*)$ with properties (1), (4) and such that

(8) $\quad a * b = b * a = s \in I$

is idempotent iff

(9) $$x * y = \text{median}(x,y,s) = \begin{cases} \min(x,y) & \text{for } x, y \leqslant s, \\ \max(x,y) & \text{for } x, y \geqslant s, \\ s & \text{otherwise.} \end{cases}$$

In particular, associative isotone idempotent binary operation $*$ has the unit iff $* = \min$ or $* = \max$.

2.6. Theorem (cf. Clifford [10]). The continuous semigroup $(I,*)$ is a direct sum of Archimedean and idempotent subsemigroups.

3. Triangular norms and conorms. Let $I = [0,1]$. Associative isotone operation $*$ is called a *triangular norm* (*t-norm*) iff it has the unit $e = 1$, and it is called a *triangular conorm* (*t-conorm*) iff it has the unit $e = 0$. t-norm (t-conorm) strictly increasing in $(0,1)^2$ is called a *strict t-norm* (t-conorm).

3.1. <u>Example (cf. Schweizer, Sklar [22])</u>. The most representative t-norms and t-conorms have specific names:

a) *lattice operations*

(10) $\quad x \wedge y = \min(x,y), \qquad\qquad x \vee y = \max(x,y),$

b) *algebraic operations*

(11) $\quad x \cdot y = xy, \qquad\qquad x \dot{+} y = x + y - xy,$

c) *bounded operations* (cf. (7))

(12) $\quad x \ominus y = \max(x+y-1, 0), \qquad\qquad x \oplus y = \min(x+y, 1).$

d) *drastic operations*

(13) $$x \underset{\cdot}{\wedge} y = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{otherwise}, \end{cases} \qquad x \dot{\vee} y = \begin{cases} x & \text{if } y = 0, \\ y & \text{if } x = 0, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in [0,1]$. Presented operations are comparable and we have

(14) $$x \underset{\cdot}{\wedge} y \leq x \ominus y \leq x \cdot y \leq x \wedge y \leq x \vee y \leq x \dot{+} y \leq x \oplus y \leq x \dot{\vee} y$$

for $x, y \in [0,1]$.

Both operations in (10)-(13) are connected by the formulas

(15) $\quad s(x,y) = 1 - p(1-x, 1-y), \quad p(x,y) = 1 - s(1-x, 1-y)$

for $x, y \in [0,1]$, where p denotes the *product operation* (t-norm), and s denotes the *sum operation* (t-conorm). From formulas (15) t-conorms can be expressed by t-norms and vice versa.

3.2. <u>Lemma (cf. Schweizer, Sklar [22])</u>. Any t-norm $*$ fulfils

(16) $\quad x \underset{\cdot}{\wedge} y \leq x * y \leq x \wedge y, \quad x * x \leq x \quad$ for $x, y \in [0,1]$.

Any t-conorm $*$ fulfils

(17) $\quad x \vee y \leq x * y \leq x \dot{\vee} y, \quad x \leq x * x \quad$ for $x, y \in [0,1]$.

The inequalities (16) and (17) are strict in $(0,1)$ for strict t-norms and t-conorms, respectively.

Using results 2.2 and 2.4 we see that continuous Archimedean t-norms and t-conorms are conjugate with suitable algebraic or bounded operation. The four characteristic operations can be used to the description of four classes of t-norms (cf. Czogała, Drewniak [9]).

3.3. <u>Theorem</u>. a) Operation $*$ is a strict continuous t-norm iff there exists an increasing bijection $h : [0,1] \to [0,1]$ such that

(18) $x * y = h^{-1}(h(x)h(y))$ for $x, y \in [0.1]$.

b) Two bijections h and k determine the same t-norm of the form (18) iff there exists a constant $c > 0$ such that (pointwise) $k = h^c$.
c) Formulas (3) and (18) determine the same t-norm iff there exists a constant $c > 0$ such that (pointwise) $h = \exp(-f/c)$ or $f = -c \log h$.

3.4. <u>Theorem</u>. a) Operation $*$ is a non-strict continuous t-norm with properties (4) and (5) iff there exists an increasing bijection $h : [0,1] \to [0,1]$ such that

(19) $x * y = h^{-1}(\max(0, h(x) + h(y) - 1))$ for $x, y \in [0.1]$.

b) The bijection h in (19) is determined uniquely by operation $*$.
c) Formulas *) (3) and (19) determine the same t-norm iff $h = 1 - f/f(0)$ or $f = -f(0)(1 - h)$.

3.5. <u>Theorem</u>. a) Operation $*$ is a strict continuous t-conorm iff there exists an increasing bijection $h : [0,1] \to [0,1]$ such that

(20) $x * y = h^{-1}(h(x) + h(y) - h(x)h(y))$ for $x, y \in [0.1]$.

b) Two bijections h and k determine the same t-conorm of the form (20) iff there exists a constant $c > 0$ such that $k = 1 - (1 - h)^c$.
c) Formulas (3) and (20) determine the same t-norm iff there exists a constant $c > 0$ such that $h = 1 - \exp(-f/c)$ or $f = -c \log(1 - h)$.

3.6. <u>Theorem</u>. a) Operation $*$ is a non-strict continuous t-conorm with properties (4) and (5) iff there exists an increasing bijection $h : [0,1] \to [0,1]$ such that

(21) $x * y = h^{-1}(\min(1, h(x) + h(y)))$ for $x, y \in [0.1]$.

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*) Formula (3) can be compared to (19) and (21) with suitable extension of f^{-1} (cf. Theorem 2.4).

b) The bijection h in (21) is determined uniquely by operation *.
c) Formulas (3) and (21) determine the same t-norm iff $h = f/f(1)$ or $f = f(1)h$.

3.7. <u>Theorem</u>. The operation * = min (* = max) is the only idempotent continuous t-norm (t-conorm).

Our consideration was restricted to continuous binary operations. A class od discontinuous binary operations in [0,1] was considered by Horiuchi [14].

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