SOFTLY DEFINED ECONOMIC MODEL

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Introduction

In this paper we introduce the concept of softly defined economic model. Moreover the concept of technological trajectories of such a model and their optimality are here presented. The starting point for our considerations is the model of economic dynamics presented by Makarov and Rubinov in [1]. In this model, as we know, an economic state at given time moment is described by means of all goods located in this time moment, i.e. by a vector with nonnegative coordinates. The transition from one economic state to another is described by a multifunction which plays the rule of the technological mapping. In our model we will assume that we have not the exact information about the quantities of goods at given time moment. We will assume that for each good we know its quantities and degrees of possibilities of these quantities. Such informations we will describe by fuzzy number. So, instead of vector with nonnegative coordinates at given time moment we will have a vector with coordinates which are fuzzy numbers. The transition from one such economic state to another will be described by F-multifunction (see [2]). The model with such informations we will call the softly defined economic model.

1. Preliminaries.

By an interval we mean a closed bounded set of "real" numbers

$$[a,b] = \{x : a \leq x \leq b\}.$$

If A is an interval, we will denote its endpoints by $\underline{A}$ and $\overline{A}$. Thus, $A = [\underline{A}, \overline{A}]$.

A convex and normal fuzzy subset in R is called a fuzzy number. Let $L(R)$ be a set of all fuzzy numbers in R such that for any $X \in L(R)$

- X is upper semicontinuous,
- supp X is compact.

From the above properties it follows that if $X \in L(R)$ then for any $\alpha \in [0,1]$, X^α is a copact interval in R, where

$$X^\alpha = \begin{cases} t : X(t) \geq \alpha & \text{if } \alpha \in (0,1], \\ t : X(t) > 0 & \text{if } \alpha = 0. \end{cases}$$

In this paper we will use the following notion

$$L^p(R) = \underbrace{L(R) \times \ldots \times L(R)}_{p \text{ times}}$$

Let $P(L^m(R))$ denotes the family of all non-void subsets of $L^m(R)$. An P-multifunction, $a : L^n(R) \to P(L^m(R))$ say, is a mapping from $L^n(R)$ to $P(L^m(R))$.
In the other words, to each element $X \in L^n(R)$ corresponds a set $a(X)$ from $P(L^m(R))$.

2. Description of the model.

Let E be a subset of R with at least two different elements. It is assumed that $0 \in E$. Elements of E will be called time moments and the element 0 initial time moment.
Let $\tilde{E} = \{ (t,\tau) \in E \times E : t < \tau \}$.

Definition 2.1. Softly defined economic model is an object

$$\mathfrak{m} = \{E,\ (L^{n_t}(R))_{t\in E},\ (X_t)_{t\in E},\ (a_{t\tau})_{(t,\tau)\in\tilde{E}}\}.$$

where

- $L^{n_t}(R) \supset K_t$ - convex cone,
- $a_{t\tau}$ - an F-multifunction, $a_{t\tau} : K_t \to P(K_\tau)$, such that
  1. superadditive,
  2. positively homogeneous,
  3. closed,
  4. bounded, (see [2]).

Moreover, it is assumed that the class $(a_{t\tau})_{(t,\tau)\in\tilde{E}}$ has the following properties: $a_{t\tau} = a_{\theta\tau} \circ a_{t\theta}$, $(t,\theta,\tau \in E,\ t<\theta<\tau)$.

Definition 2.2. A technological trajectory of $\mathfrak{m}$ is a family $Tr = (X_t)_{t\in E}$ such that:

- $X_t \in K_t$, $(t \in E)$,
- $X_\tau \in a_{t\tau}(X_t)$, $((t,\tau) \in \tilde{E})$.

In this case X_t is called the state of trajectory Tr at the time t, X_0 is the initial state of Tr. It is said that the trajectory Tr goes out X if $X=X_0$ and passes through X at the time moment t if the state of Tr at the time moment t is X.

Theorem 2.1 (Existence of technological trajectory). Let $X_0 \in K_0$ and $Y' \in a_{0t'}(X_0)$. Then there exists a technological trajectory Tr of $\mathfrak{m}$ going out X_0 and passing through Y' at the moment of time t'.

Proof. For every $t \in E$ let us take element $W_t \in L^{n_t}(R)$ such that $W_t \notin K_t$ and let us denote $L = \prod_{t\in E} L_t$, where $L_t = K_t \cup \{W_t\}$. Next let us choose a subset $M \subset L$, of elements of Tr such that:

- there exists a subset $E_{Tr} \subset E$ such that
  a) $0 \in E_{Tr}$, $t' \in E_{Tr}$,
  b) if $t,\tau \in E_{Tr}$, $t<\tau$, then $X_\tau \in a_{t\tau}(X_t)$,

c) $X_0 = X_0$, $X_1 = Y^*$,

d) if $t \in E \setminus E_{Tr}$ then $X_t = W_t$.

By the way, let us mention that $M \neq \emptyset$. In M one can introduce a partial ordering $\succcurlyeq$ as follows:

$$Tr_1 \succcurlyeq Tr_2 \qquad (Tr_1 = (X_t^1)_{t \in E} \in M,\ Tr_2 = (X_t^2)_{t \in E} \in M)$$

iff

1) $E_{Tr_1} \supset E_{Tr_2}$,

2) $X_t^1 = X_t^2 \quad (t \in E_{Tr_2})$.

Now, it will be checked that each chain in M is bounded from above. Let $(Tr_\alpha)_{\alpha \in A}$ denote a chain in M, $Tr_\alpha = (X_t^\alpha)_{t \in E}$, $\alpha \in A$.

It is seen that the element $Tr = (X_t)_{t \in E}$ such that

- $X_t = X_t^\alpha$ if $t \in E_{Tr}$

- $X_t = W_t$ if $t \notin \bigcup_{\alpha \in A} E_{Tr_\alpha}$

belongs to M and $E_{Tr} = \bigcup_{\alpha \in A} E_{Tr_\alpha}$. Moreover, for every $\alpha \in A$, $Tr \succcurlyeq Tr_\alpha$. So, the chain $(Tr_\alpha)_{\alpha \in A}$ is bounded from above, as asserted. By Zorn's lemma the set M has a maximal element. It remains now to prove that for each maximal element Tr of M there holds $E_{Tr} = E$. Indeed, let us suppose that there exists a maximal element $Tr \in M$ such that $E \setminus E_{Tr} \neq \emptyset$. So, there exists an element $\theta \in E \setminus E_{Tr}$. Now, let us define

$$F_1 = \{t \in E_{Tr} : t < \theta\},\quad F_2 = \{\tau \in E_{Tr} : \theta < \tau\},\quad F = F_1 \times F_2$$

and for $(t, \tau) \in F$

$$b_{t\tau} = a_{\theta\tau}^{-}(X_\tau) \cap a_{t\theta}(X_t).$$

(1) The subset $b_{t\tau}$ is a non-empty subset.

In fact, if $t < \theta < \tau$ and $X_\tau \in a_{t\tau}(X_t) = a_{\theta\tau} \circ a_{t\theta}(X_t)$ then there exists an element $Y \in K_\theta$ such that $Y \in a_{t\theta}(X_t)$, $X_\tau \in a_{\theta\tau}(Y)$. From this inclusion it follows that $Y \in b_{t\tau}$ i.e. $b_{t\tau} \neq \emptyset$.

(2) The set $b_{t\tau}$ is a compact set.

The F-multifunction a_{t0} is closed and bounded, so the set $a_{t0}(X_t)$ is compact. Moreover $a^{\leftarrow}_{0\tau}(X_\tau)$ is closed. So, the set $b_{t\tau}$ is an intersection of two sets, a closed and a compact one. Therefore $b_{t\tau}$ is a compact set.

(3) If $(\tau, t_1) \in F$, $(\tau, t_2) \in F$ and $t_1 > t_2$, then $b_{\tau t_1} \subset b_{\tau t_2}$.

For $t_2 < t_1 < 0$, we have

$$a_{t_2 0}(X_{t_2}) = a_{t_1 0} \circ a_{t_2 t_1}(X_{t_2}).$$

This means that $a_{t_2 0}(X_{t_2}) \supset a_{t_1 0}(X_{t_1})$. From this it follows that $b_{\tau t_1} \subset b_{\tau t_2}$.

(4) If $(\tau_1, t) \in F$, $(\tau_2, t) \in F$ and $\tau_1 > \tau_2$ then $b_{\tau_1 t} \supset b_{\tau_2 t}$.

From (1), (3) and (4) it follows that the family $(b_{t\tau})_{(t,\tau)\in F}$ is centered. Let $(t_i, \tau_i) \in F$ $(i=1,\dots,n)$ and $\tau = \min_i \tau_i$, $t = \max_i t_i$.

It is seen that $(t, \tau) \in F$. Moreover $b_{t\tau} \subset b_{t_i \tau} \subset b_{t_i \tau_i}$ $(i=1,\dots,n)$.

So, $b_{t\tau} \subset \bigcap_{i=1}^{n} b_{t_i \tau_i}$. But $b_{t\tau} \neq \emptyset$, therefore $\bigcap_{i=1}^{n} b_{t_i \tau_i} \neq \emptyset$.

Compactness of $b_{t\tau}$ and centerness of the family $(b_{t\tau})_{(t,\tau)\in F}$ yields $\bigcap_{(t,\tau)\in F} b_{t\tau} \neq \emptyset$. Let $X_0 \in \bigcap_{(t,\tau)\in F} b_{t\tau}$ and let us take into account an element $\overline{W} = (\bar{X}_t)_{t\in E}$ such that

$$\bar{X}_t = \begin{cases} X_t, & t \in E_W, \\ X_0, & t = 0, \\ W_t, & t \in E \setminus (E_W \cup \{0\}). \end{cases}$$

Element $\overline{W} \in M$ and $E_{\overline{W}} = E_W \cup \{0\}$. So, we get $\overline{W} \succeq W$ and $\overline{W} \neq W$ in contradiction with the supposed that W is the maximal element in M.

3. Problem of the optimality.

Now, let us additionally assume that there exists $T \in E$ such that $t \leq T$ for all $t \in E$.
Let U^α denote the set $\{ X^\alpha ; X \in a_{0T}(X_0) \}$.

Definition 3.1. A technological trajectory Tr with initial state X_0 and terminal state X_T is called optimal if there exists a non-zero linear functional p such that for any $\alpha \in [0,1]$

$$p(X_T^\alpha) = \max_{y \in U^\alpha} p(y) > 0 .$$

Let A denote an arbitrary set such that $A \neq \emptyset, \{0\}$. An element z of A is called the limiting point from above of A if $a \cdot z \notin A$ for $a > 1$. For a normal covering of A we use the symbol nA (compare e.g. [1]).

Theorem 3.1. A technological trajectory Tr with initial state X_0 and terminal state X_T is optimal iff for all $\alpha \in [0,1]$, X_T^α is a limiting point from above of nU^α.

Proof. Let Tr denote an optimal trajectory with initial and terminal states X_0 and X_T respectively. We will prove that X_T^α is a limiting point from above of nU^α. In contrary, suppose there exists an $a > 1$ such that $a \cdot X_T^\alpha \in nU^\alpha$. Because $p(X_T^\alpha) > 0$ we get

$$p(X_T^\alpha) = \max_{y \in U^\alpha} p(y) = \max_{y \in nU^\alpha} p(y) \geq p(a \cdot X_T^\alpha) = a \cdot p(X_T^\alpha) > 0,$$

i.e. $1 \geq a$ in contradiction with $a > 1$.

Now, let us assume that X_T^α is a limiting point from above of the set nU^α ($\alpha \in [0,1]$). Let S^α denote the sphere $nU^\alpha - nU^\alpha$ and $\| \cdot \|_{nU^\alpha}$ Minkowski's norm. It is known (e.g. [1]) that this norm is monotonous and

$$nU^\alpha = \{ z ; \| z \|_{nU^\alpha} \leq 1 \} .$$

Because X_T^{α} is a limiting point from above of the set nU^{α}, there holds

$$\| X_T \|_{nU^{\alpha}} = 1.$$

Therefore, there exists a non-zero, linear functional p such that

$$p(X_T^{\alpha}) = \| X_T^{\alpha} \|_{nU^{\alpha}} = 1, \quad \|p\| = 1.$$

So, it is proved that for the trajectory Tr the condition of the optimality is fulfilled and thus the proof is finished.

A direct consequent of this proof is :

Theorem 3.2. A technological trajectory Tr with initial and terminal states X_0 and X_T resp. is optimal iff $\| X_T^{\alpha} \|_{nU^{\alpha}} = 1$ for all $\alpha \in [0,1]$.

References

[1] V.L.Makarov and A.M.Rubinov, Mathematical Theory of Economic Dynamics and Equilibrium (Moscow, 1973) (in Russian).

[2] M.Matłoka, Some properties of F-multifunctions, Proc.XI Circle Mathematical Session, Poznań 1987.