SOME REMARKS ON THE ROBUSTNESS OF LL-TYPE FUZZY LINEAR ALGEBRAIC SYSTEMS.

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The notion of membership function is the basic and elementary object of fuzzy sets theory. There are different methods for obtaining membership functions (see, for example, [2], [4]), but since an elementary object of some theory "must be set outside the theory and therefore its adequacy can not be checked by tools of this theory" ([2]), all of them use some information from outside the theory. The distinctive feature of fuzzy sets theory is the fact, that, as a rule, this outside information explicity or implicity is of a subjective character, and in most cases is obtained from experts or even from human intuition and experience. This being the case, this information may be sensitive to different factors; therefore membership functions are equally sensitive. That is why the problem of the sensivity of results, which are obtained by methods that employ fuzzy sets theory, to the knowledge of the initial membership functions becomes very actual.

In this work we will discuss one aspect of a wide set of questions which are arised by this important problem, that of, the solution of LL-type fuzzy systems of linear algebraic equations.

Let us consider the following system:

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 + \dots + \tilde{a}_{1n}\tilde{x}_n = \tilde{b}_1 \\ \dots\dots \\ \tilde{a}_{n1}\tilde{x}_1 + \dots + \tilde{a}_{nn}\tilde{x}_n = \tilde{b}_n \end{cases} \tag{1}$$

where $\tilde{a}_{ij}=(a_{ij},\alpha_{ij},\beta_{ij})_{LL}$, $\tilde{b}_i=(b_i,\underline{b}_i,\overline{b}_i)_{LL}$ are LL-type fuzzy numbers.

(the reader can find detailed information about LL-type or more generally LR-type fuzzy numbers in the works of Dubois and Prade, for example, in [3], [4])

One possible approach to the solution of LL-type linear fuzzy system (1) was introduced in [1] . The natural suggeston that $\tilde{x}_i$ also belongs to the class of LL-type fuzzy numbers was made (i.e. $\tilde{x}_i=(x_i,p_i,q_i)_{LL}$), and as a result of using methods proposed in [1] such LL-type fuzzy linear system may in some cases be reduced to the following non-fuzzy one:

$$\begin{cases} A_i = b_i \\ A_i^{*} = \underline{b}_i + \overline{b}_i \qquad (i=\overline{1,n}) \\ A_i^{**} = \underline{b}_i - \overline{b}_i \end{cases} \qquad (2)$$

where
$$A_i = \sum_{j=1}^{n} a_{ij}x_j$$
$$A_i^{*} = \sum_{j=1}^{n} (|x_j|(\alpha_{ij}+\beta_{ij})+|a_{ij}|(p_j+q_j)) \qquad (3)$$
$$A_i^{**} = \sum_{j=1}^{n} (x_j(\alpha_{ij}-\beta_{ij})+a_{ij}(p_j-q_j))$$

In general, however, (1) is reduced to the following non-fuzzy non-linear optimization problem:

$$\begin{cases} \min\limits_{x_j,p_j,q_j} F(A_1-b_1,\dots,A_n-b_n,A_1^{*}-(\underline{b}_1+\overline{b}_1),\dots,A_n^{*}-(\underline{b}_n+\overline{b}_n),\dots,A_n^{**}-(\underline{b}_n-\overline{b}_n)) \\ (x_j-L_+^{-1}(0)p_j)(x_j+L_+^{-1}(0)q_j) \geqslant 0 \\ p_j \geqslant 0 \qquad (j=\overline{1,n}) \\ q_j \geqslant 0 \end{cases} \qquad (4)$$

where F is a certain functional, measuring the deviatation of the left-hand side of (2) from the right-hand one; A_i, A_i^{*}, A_i^{**} are given by (3) and $L_+^{-1}(0)$ is the minimal positive root of the equation $L(x)=0$ (its existence and finitness were shown in [1]).

Finding the values of x_j, p_j, q_j $(j=\overline{1,n})$ as the solution of (4), we obtain the solution of (1) : $\tilde{x}_j=(x_j,p_j,q_j)_{LL}$.

It is important to raise the problem of robastness of the solution of (1) under changing of coefficients and right-hand sides of (1) (by "changing" we mean changing their membership functions). Would little changes of parameters of (1) lead to significant changes of its solution or not? In other words, would the solution of (1) be stable, if the coefficients be changed?

Consider the following problem.

Let $\mathcal{L}_t = \{L_t(x)=\max(0;1-|x|^t), 0 \le t < +\infty\}$, where t is a parameter.

Let $\tilde{a} = (a,\alpha,\beta)_{L_tL_t}$. Figures 1a - 1e show that from $\mathcal{L}_t$ one may obtain different types of fuzzy numbers:

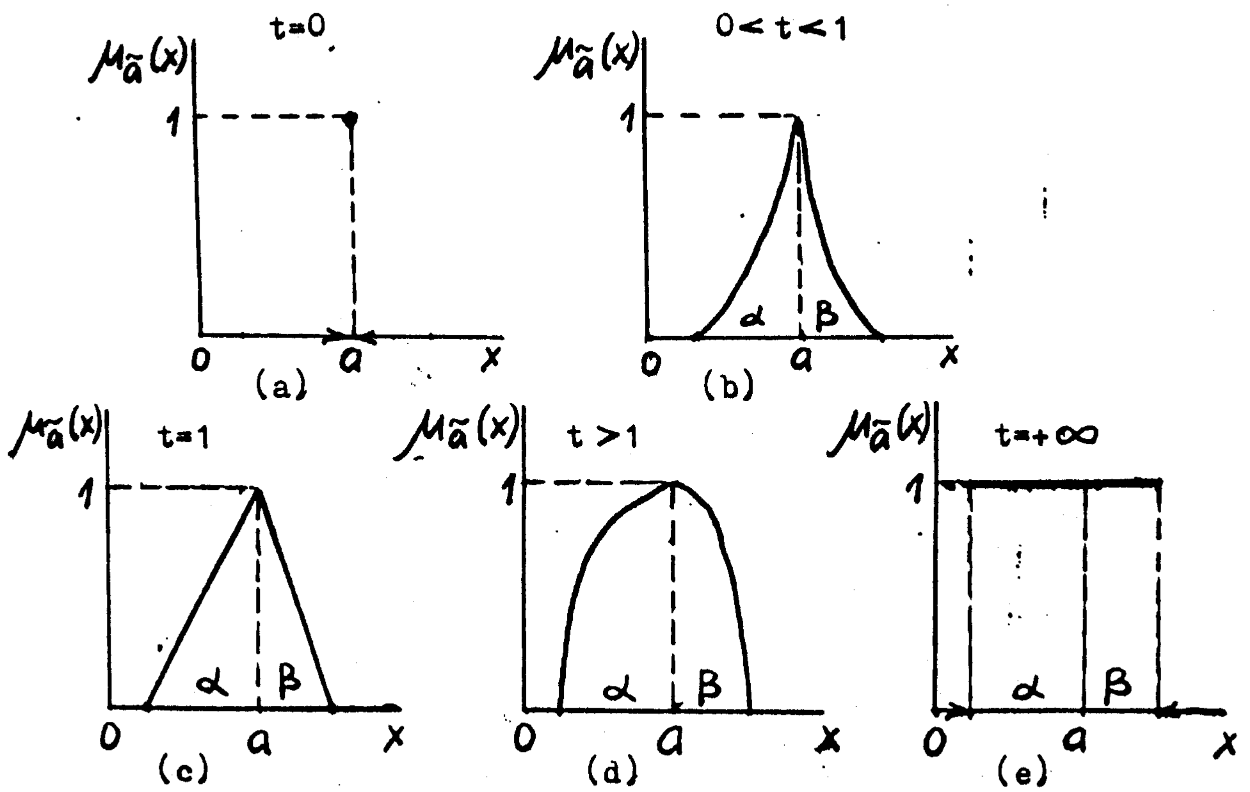

Figure 1. Different types of LL-type fuzzy numbers obtained from $\mathcal{L}_t$.

Remark 1. $\forall t \geqslant 0,\ L_t(x) \in \mathcal{L}_t$

$L_{+t}^{-1}(0) = \text{const} = 1$

Remark 2. $\forall t \geqslant 0,\ L_t(x) \in \mathcal{L}_t$

$\operatorname{supp} \tilde{a}_{ij} = [a_{ij} - \alpha_{ij}; a_{ij} + \beta_{ij}]$; $\operatorname{supp} \tilde{b}_i = [b_i - \underline{b}_i; b_i + \overline{b}_i]$

and they do not depend on the choice of t (or $L_t(x) \in \mathcal{L}_t$).
(remember that for $\tilde{a} = (a, \alpha, \beta)_{LL}$ $\operatorname{supp} \tilde{a} = [a - L_+^{-1}(0)\alpha; a + L_+^{-1}(0)\beta]$)

Let $t_0 \geqslant 0$ $L_{t_0}(x) = \max(0; 1 - |x|^{t_0}) \in \mathcal{L}_t$ and consider the respective $L_{t_0}L_{t_0}$-type fuzzy linear system, i.e. assume $L = L_{t_0}$ in (1). Let the solution of (1), obtained by solving (4) denoted as $\tilde{x}_j^0 = (x_j^0, p_j^0, q_j^0)_{L_{t_0}L_{t_0}}$ $(j = \overline{1,n})$.

Now let us take $t_1 \geqslant 0$, $t_1 \neq t_0$, $L_{t_1}(x) = \max(0; 1 - |x|^{t_1}) \in \mathcal{L}_t$. What will happen to the solution of (1) when $L = L_{t_1}$, i.e. when (1) is transformed from $L_{t_0}L_{t_0}$-type fuzzy linear system into $L_{t_1}L_{t_1}$-type one?

In other words we are discussing the following problem:

Consider that we know exactly the mean values and the boundaries of the changing of coefficients and right-hand sides of linear system, i.e. we know precise values of $a_{ij}, \alpha_{ij}, \beta_{ij}, b_i, \underline{b}_i, \overline{b}_i$ in (1) $(i = \overline{1,n};\ j = \overline{1,n})$, but we do not possess exact knowledge about membership functions of $\tilde{a}_{ij}$ and $\tilde{b}_i$ inside their supports; we only know (or suppose) that they belong to $\mathcal{L}_t$. How will the solution of the linear system depend on the choice of $L_t(x) \in \mathcal{L}_t$?

Formally we must find the value of

$$\Delta\mu^*_{\tilde{x}_j} = \sup_{x \in \operatorname{supp} \tilde{x}_j} |\Delta\mu_{\tilde{x}_j}(x)| = \sup_{x \in \operatorname{supp} \tilde{x}_j} |\mu_{\tilde{x}_j^1} - \mu_{\tilde{x}_j^0}| \qquad (5)$$

Remark 3. The values of x_j, p_j, q_j $(j = \overline{1,n})$, obtained by solving (4), involve the type of L-function only via the value $L_+^{-1}(0)$.

Remark 1 and Remark 3 lead us to the important result:

Proposition 1. $\forall\, t_0 \geqslant 0,\ t_1 \geqslant 0,\ L_{t_0}(x) \in \mathcal{L}_t,\ L_{t_1}(x) \in \mathcal{L}_t$

$$x_j^0 = x_j^1 = x_j;\ p_j^0 = p_j^1 = p_j;\ q_j^0 = q_j^1 = q_j \qquad (6)$$

(more generally this Proposition may be extended to an arbitrary set of L-functions $\mathcal{L}_t$ if $L^{-1}_{+t}(0)$=const holds for all t) -

Remark 4. $\forall\, t > 0,\ L_t(x) \in \mathcal{L}_t$

$$\operatorname{supp} \tilde{x}_j = [x_j - p_j; x_j + q_j]$$

and it does not depend on the choice of t (or $L_t(x) \in \mathcal{L}_t$).

Remark 5. If t_0 or t_1 is equal to 0 or infinite, it is clear that $\Delta\mu^*_{\tilde{x}_j} = 1$.

So in the following calculations we suppose that t_0 and t_1 are both positive and finite. Then $\mu_{\tilde{x}_j^0}$ and $\mu_{\tilde{x}_j^1}$ are continious functions (see Fig 1b - 1d) and we have:

$$\Delta\mu^*_{\tilde{x}_j} = \sup_{x \in \operatorname{supp} \tilde{x}_j} |\Delta\mu_{\tilde{x}_j}(x)| = \sup_{x \in \operatorname{supp} \tilde{x}_j} |\mu_{\tilde{x}_j^1} - \mu_{\tilde{x}_j^0}| =$$

$$= \max_{x \in [x_j - p_j; x_j + q_j]} |\mu_{\tilde{x}_j^1}(x) - \mu_{\tilde{x}_j^0}(x)| =$$

$$= \max\Big(\max_{x \in [x_j - p_j; x_j]} \Big|L_{t_1}\Big(\frac{x_j - x}{p_j}\Big) - L_{t_0}\Big(\frac{x_j - x}{p_j}\Big)\Big|,\ \max_{x \in [x_j; x_j + q_j]} \Big|L_{t_1}\Big(\frac{x - x_j}{q_j}\Big) - L_{t_0}\Big(\frac{x - x_j}{q_j}\Big)\Big|\Big) =$$

$$= \max_{z \in [-1;1]} |L_{t_1}(z) - L_{t_0}(z)| = \max_{z \in [-1;1]} \big||z|^{t_0} - |z|^{t_1}\big| =$$

$$= \max_{z \in [0;1]} |z^{t_0} - z^{t_1}| \qquad (7)$$

Remark 6. $\Delta\mu^*_{\tilde{x}_j}$=const for all j.

Let $t_0 \geqslant t_1$. Then $\Delta\mu^*_{\tilde{x}_j} = \max_{z \in [0;1]} z^{t_0}(z^{\Delta t} - 1)$ where $\Delta t = t_1 - t_0 \leqslant 0$.

It is easy to show that in this case

$$\Delta\mu^*_{\tilde{x}_j} = \Big(\frac{t_0}{t_1}\Big)^{\frac{t_0}{\Delta t}} \Big(\frac{t_0}{t_1} - 1\Big) \qquad (8)$$

For the case $t_1 > t_o$ $(\Delta t > 0)$ we have:

$$\Delta\mu^{*}_{\tilde{x}_j} = \left(\frac{t_o}{t_1}\right)^{\frac{t_o}{\Delta t}}\left(1 - \frac{t_o}{t_1}\right) \tag{9}$$

Together (8) and (9) will give us the following result:

$$\Delta\mu^{*}_{\tilde{x}_j} = \left(\frac{t_o}{t_1}\right)\left|1 - \frac{t_o}{t_1}\right| = \left(\frac{1}{1+\varepsilon_{t_o}}\right)^{\frac{1}{\varepsilon_{t_o}}}\frac{|\varepsilon_{t_o}|}{1+\varepsilon_{t_o}} = |\varepsilon_{t_o}|(1+\varepsilon_{t_o})^{-(1+\frac{1}{\varepsilon_{t_o}})} \tag{10}$$

where $\varepsilon_{t_o} = \frac{\Delta t}{t_o} = \frac{t_1 - t_o}{t_o} > -1$ (since $t_1 > 0$) is a relative error of t_o.

If to(10) we add the result of Remark 5 we have:

$$\Delta\mu^{*}_{\tilde{x}_j} = \begin{cases} 1, & \varepsilon_{t_o} = -1 \\ |\varepsilon_{t_o}|(1+\varepsilon_{t_o})^{-(1+\frac{1}{\varepsilon_{t_o}})}, & -1 < \varepsilon_{t_o} < +\infty \\ 1, & \varepsilon_{t_o} = +\infty \end{cases} \tag{11}$$

<u>Remark 7.</u> $\Delta\mu^{*}_{\tilde{x}_j}$ does not depend on j, nor on n, nor on the absolute value of t_o(!).

Table 1 shows the values of $\Delta\mu^{*}_{\tilde{x}_j}$ depending on different values of ε_{t_o}:

<u>Table 1</u>

ε_{t_o}	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4
$\Delta\mu^{*}_{\tilde{x}_j}$	1.0000	0.6968	0.5350	0.4178	0.3257	0.2500	0.1859
ε_{t_o}	-0.3	-0.2	-0.1	-0.05	-0.025	-0.01	0.00
$\Delta\mu^{*}_{\tilde{x}_j}$	0.1305	0.0812	0.0387	0.0189	0.0093	0.0037	0.0000
ε_{t_o}	0.01	0.025	0.05	0.1	0.2	0.3	0.4
$\Delta\mu^{*}_{\tilde{x}_j}$	0.0036	0.0091	0.0179	0.0351	0.0670	0.0962	0.1232
ε_{t_o}	0.5	0.6	0.7	0.8	0.9	1.0	$+\infty$
$\Delta\mu^{*}_{\tilde{x}_j}$	0.1481	0.1713	0.1929	0.2132	0.2321	0.2500	1.0000

Table 1 leads us to the conclusion that the solution of an LL-type fuzzy linear system of algebraic equations in case $L \in \mathcal{L}_t$ is robust, that is, little changes of L-function lead to wholly insignificant changes of the solution (in practice, as a rule, $|\mathcal{E}_t| \le$ 10-20% (i.e. $|\mathcal{E}_t| \le 0.1-0.2$) because in the opposite case it means that the researcher has too little information about the problem he is studying).

We are now able to solve, for example, the following problem:

Given the value of the relative error ±5%, how sensitive will the solution of (1) be towards such changes of L-function? We can now answer such a question:

$$\Delta\mu^{*}_{\tilde{x}_j} = \sup_{t_1, t_2 : \left|\frac{t_2 - t_1}{t_1}\right| \le 0.05} \ \sup_{x \in \mathrm{supp}\, \tilde{x}_j} \left| \mu_{\tilde{x}_j}^{t_1} - \mu_{\tilde{x}_j}^{t_2} \right| =$$

$$= \max\,(0.0189; 0.0179) = 0.0189$$

Remember that $\Delta\mu_{\tilde{x}_j}(x_j - p_j) = \Delta\mu_{\tilde{x}_j}(x_j) = \Delta\mu_{\tilde{x}_j}(x_j + q_j) = 0$ and that $\Delta\mu^{*}_{\tilde{x}_j}$ does not depend on the absolute value of t.

We can also consider an inverse problem:

Let us assume $\Delta\mu^{*}_{\tilde{x}_j} = 0.040$. What interval of change for t is available in this case?

Using Table 1 we can find an approximate answer. The relative error of changing t must be not greater than 0.1 or 10%.

Conclusion.

This work is devoted only to one particular case of the important problem of robastness of LL-type fizzy linear algebraic systems. We have presumed that mean values and boundaries of coefficients and right-hand sides remain constant, and only the type of L-function varies inside the set $\mathcal{L}_t$.

Obviously, consideration of cases where these parameteres are also varied is connected with the problem of the stability of a non-fuzzy non-linear optimization problem (4), but that is beyond the subject of this work.

References.

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