ON THE FUZZY TR–REGRESSION [1]

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Preliminaries

Let X, Y be classical sets of objects. A fuzzy subset A of X is defined by its membership function $\mu_A : X \to [0,1]$. The height of A is $\mathrm{hgt}(A) = \sup\{\mu_A(x) : x \in X\}$. The set of fuzzy subsets of X will be denoted by $\mathcal{F}(X)$. Two fuzzy sets A and B of X are equal $(A = B)$ if and only if $\mu_A(x) = \mu_B(x)$ $(\forall x \in X)$. A is said to be included in B $(A \subset B)$ if $\mu_A(x) \le \mu_B(x)$ $(\forall x \in X)$. In this sense, the classical subsets of X are identified with special fuzzy sets, e.g. the membership function of the classical sets $\emptyset, A, X$ are, respectively, $\mu_\emptyset(x) = 0$, $\mu_A(x) = \chi_A(x)$, $\mu_X(x) = 1$ $(\forall x \in X)$.

In fuzzy theory a general intersection (union) operator can be given by a triangular norm (t–conorm). A function T from $[0,1] \times [0,1]$ to $[0,1]$ will be called t–norm if $\forall u, v, z \in [0,1]$ $T(u,v) = T(v,u)$; $T(T(u,v),z)) = T(u,T(v,z))$; $T(u,v) \le T(u,z)$ if $v \le z$; $T(u,1) = u$. T is a strong t–norm if T is a continuous and strong monotone function. S is a (strong) t–conorm if the function $T_S(u,v) = 1 - S(1-u, 1-v)$ is a (strong) t–norm. The T–intersection and S–union of $A, B \in \mathcal{F}(X)$ are fuzzy sets of $\mathcal{F}(X)$ with membership functions

$$\mu_{A\cap_T B}(x) = T(\mu_A(x), \mu_B(x)); \qquad \mu_{A\cup_S B}(x) = S(\mu_A(x), \mu_B(x)), \quad (\forall x \in X).$$

The membership function of the complement of A is $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$. A T–direct product of $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$ denoted by $A \times_T B$ is a fuzzy subset of $X \times Y$ defined by $\mu_{A\times_T B}(x,y) = T(\mu_A(x), \mu_B(y))$. We will use the following special t–norms and conorms:

$$T(u,v) = u \wedge v = \min(u,v) \qquad S(u,v) = u \vee v = \max(u,v);$$
$$T(u,v) = u \odot v = uv \qquad S(u,v) = u \oplus v = u + v - uv;$$
$$T(u,v) = u \sqcap v = \max(0, u+v-1)) \qquad S(u,v) = u \sqcup v = \min(1, u+v);$$

$$T_w(u,v) = \begin{cases} \min(u,v) & \text{if } \max(u,v) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

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A fuzzy set $R \in \mathcal{F}(X \times Y)$ is called a fuzzy relation in $X \times Y$. The projection of R on X is a fuzzy set $R_X \in \mathcal{F}(X)$ defined by $\mu_{R_X}(x) = \sup\{\mu_R(x,y) : y \in Y\}$. The cylindrical extension of $F \in \mathcal{F}(X)$ is a fuzzy set $F^x \in \mathcal{F}(X \times Y)$ so that $\mu_{F^x}(x,y) = \mu_F(x), \quad \forall (x,y) \in X \times Y$. A fuzzy set (relation) $R \in \mathcal{F}(X \times Y)$ is said to be T-separable if R is a fuzzy T-direct product set, i.e. $\exists A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$, so that $R = A \times_T B$.

Fuzzy TR-regression

Our approach to fuzzy regression problem is more theoretical than practical. The idea of this concept originates from the classical case and is based on the substitution of notions of probability theory used in classical regression with fuzzy ones. In place of joint density function we take fuzzy relation and from this we derive the so-called fuzzy TR-regression functions similarly to the classical process of defining the conditional density function. In the next step we give set-valued functions (εTR-regressions) and an unambigous set-valued function (regression) which is analogous to the classical theoretical regression, i.e. to the conditional expected value.

Let $\emptyset \neq R \in \mathcal{F}(\mathbf{R}^N \times \mathbf{R}^k)$ be a given fuzzy relation on $\mathbf{R}^n \times \mathbf{R}^k$,

$$X = \{x \in \mathbf{R}^n : \exists y \in \mathbf{R}^k, \mu_R(x,y) > 0\},$$

$$Y = \{y \in \mathbf{R}^k : \exists x \in \mathbf{R}^n, \mu_R(x,y) > 0\}.$$

The following studies will be restricted to $X \times Y$ and R will be regarded as a fuzzy relation $R \in \mathcal{F}(X \times Y)$. Moreover, let T be a fixed t-norm.

In the definition of fuzzy TR-regression we generalize the well-known connection between joint and conditional density functions and the density function of explanatory variable(s): $f(x,y) = f(x) \cdot f(y|x)$.

Definition. A fuzzy set (relation) $r \in \mathcal{F}(X \times Y)$ is called a fuzzy TR-regression if $R = (R_X)^x \cap_T r$, i.e.

$$\mu_R(x,y) = T(\mu_{R_X}(x), \mu_r(x,y)) \qquad \forall (x,y) \in X \times Y.$$

The set of fuzzy TR-regressions will be denoted by $\mathcal{F}_{TR} = \mathcal{F}_{TR}(X \times Y)$.

In the following we give some basic theorems and properties.

Theorem.

(i) T is continuous $\Longrightarrow$ $\mathfrak{F}_{TR} \neq \emptyset$ and $\mathfrak{F}_{TR} = \{ r \in \mathcal{F}(X \times Y) : r_{\min} \subset r \subset r_{\max} \}$ (i.e. a fuzzy TR–regression exists);

(ii) T is strong $\Longrightarrow$ $|\mathfrak{F}_{TR}| = 1$ (i.e. we have unambigous fuzzy TR–regression);

(iii) $R_X \equiv X$ $\Longrightarrow$ $\mathfrak{F}_{TR} = \{R\}$ (i.e. R is the unique fuzzy TR–regression);

(iv) $\mathfrak{F}_{TR} \neq \emptyset$ $\Longrightarrow$ $R \vee r_{\min} \subset r \subset r_{\max} \quad \forall r \in \mathfrak{F}_{TR}$,

where $r_{\min}, r_{\max} \in \mathcal{F}(X \times Y)$ are defined for $\mathfrak{F}_{TR} \neq \emptyset$ by

$$\mu_{r_{\min}}(x,y) = \inf_{r \in \mathfrak{F}_{TR}} \mu_r(x,y); \quad \mu_{r_{\max}}(x,y) = \sup_{r \in \mathfrak{F}_{TR}} \mu_r(x,y).$$

Using special fuzzy intersection operators we have the following special cases:

When $T = \wedge$ then $r_{\min} = R$, $r_{\max} = R \vee \{ \mu_R = \mu_{(R_X)^z} \}$.
When $T = \odot$ then r, defined by $\mu_r = \frac{\mu_R}{\mu_{(R_X)^z}}$, is the unique fuzzy TR–regression.
When $T = \sqcap$ then $r_{\max} = \overline{(R_X)^z} \sqcup R$, $r_{\min} = r_{\max} \sqcap \{\mu_R > 0\}$.
When $T = T_w$ then it may be $\mathfrak{F}_{TR} = \emptyset$ (T_w is not continuous!), for example for a fuzzy relation $R \in \mathcal{F}(X \times Y)$ which satisfies $0 < \mu_R(x,y) < \mu_{R_X}(x) \quad \forall (x,y) \in X \times Y$.
In the following we assume that T is continuous.

Theorem. Let $R \in \mathcal{F}(X \times Y)$ be a T–separable fuzzy relation, $R = F \times_T G = F^z \cap_T G^z$ $(F \in \mathcal{F}(X),\ G \in \mathcal{F}(Y))$. In this case

(i) $\exists r \in \mathcal{F}(Y)$ so that $r^z \in \mathfrak{F}_{TR}$;

(ii) when $\mathrm{hgt}(G) = 1$ then $G^z \in \mathfrak{F}_{TR}$.

This theorem expresses that in case of separability we also have cylindrical–type fuzzy TR–regression. A separable fuzzy relation is, consequently, analogous to the joint density function of independent random variables. In this sense the fuzzy extension of regression with independent explanatory variables can be formulated as follows:

Theorem. Assuming that $X = X_1 \times \cdots \times X_n$, $R_i \in \mathcal{F}(X_i \times Y)$, $R = (R_1)^z \cap_T \cdots \cap_T (R_n)^z$. If $r_i \in \mathfrak{F}_{TR}(X_i \times Y)$ $(i = 1, \ldots, n)$ then $r = (r_1)^z \cap_T \cdots \cap_T (r_n)^z \in \mathfrak{F}_{TR}(X \times Y)$, where $(R_i)^z$ and $(r_i)^z$ are cylindrical extensions of R_i and r_i on $X \times Y$.

Based on a fuzzy TR–regression we can define the following family of set–valued functions:

Definition. Let $r \in \mathcal{F}_{TR}$ be a given fuzzy TR-regression and $\varepsilon \in [0,1]$ fixed. The function

$$f_{r\varepsilon}: X \longrightarrow \mathcal{P}(Y) = \{ A : A \subset Y \}, \quad x \longmapsto \{ y \in Y : \mu_r(x,y) \geq \varepsilon \}$$

is called a (set valued) εTR-regression (generated by r).

The following properties of $f_{r\varepsilon}$ hold for all $x \in X$:

(a) $\varepsilon = 0 \quad \Longrightarrow \quad f_{r0} \equiv Y;$
$\varepsilon = 1 \quad \Longrightarrow \quad f_{r1} \subset f_{r\varepsilon}(x) \quad \forall \varepsilon < 1;$

(b) $\varepsilon_1 < \varepsilon_2 \quad \Longrightarrow \quad f_{r\varepsilon_2}(x) \subset f_{r\varepsilon_1}(x);$

(c) $f_{r_{\min}\varepsilon}(x) \subset f_{r\varepsilon}(x) \subset f_{r_{\max}\varepsilon}(x);$

(d) When R is T-separable and $r^x \in \mathcal{F}_{TR}$ then $f_{r\varepsilon}(x) = \{ y \in Y : \mu_r(y) \geq \varepsilon \};$

(e) $f_{r_{\max}1}(x) = \{ y \in Y : \mu_R(x,y) = \mu_{R_X}(x) \}$ independent of the t–norm.

Based on these properties we define the set–valued regression as follows:

Definition. Let $R \in \mathcal{F}(X \times Y)$ be given. The function $f_R = f_{r_{\max}1}$ is called a set–valued regression .

Examples

A. Let $g: \mathbf{R}^n \longrightarrow \mathbf{R}$ be a fixed function, $X =$Dom g, $Y =$Im g and $R = R_g$, where

$$\mu_{R_g}(x,y) = \begin{cases} 1 & \text{if } y = g(x); \\ 0 & \text{otherwise.} \end{cases}$$

In this case $R_X \equiv X$ so that $r \equiv R$ and $f_{r\varepsilon}(x) = f_R(x) = \{g(x)\}$.

B. Let $g, \sigma: \mathbf{R}^n \longrightarrow \mathbf{R}$ be fixed functions, $s = \inf\{\sigma(x) : x \in \mathbf{R}^n\} > 0$, and $R \in \mathcal{F}(\mathbf{R}^n \times \mathbf{R})$ defined by the membership function

$$\mu_R(x,y) = \frac{s \cdot e^{-\frac{(y-g(x))^2}{\sigma^2(x)}}}{\sigma(x)}.$$

– If $T = \odot$ then $\mu_{R_X}(x) = \frac{s}{\sigma(x)}$, and the fuzzy TR–regression is given by

$$\mu_r(x,y) = e^{-\frac{(y-g(x))^2}{\sigma^2(x)}}.$$

From this one has:

$$f_{r\varepsilon}(x) = \left\{ y \in \mathbf{R} : \ g(x) - \sigma(x) \cdot \sqrt{\ln \frac{1}{\varepsilon}} \leq y \leq g(x) + \sigma(x) \cdot \sqrt{\ln \frac{1}{\varepsilon}} \ \right\}$$

and, finally, $f_R(x) = \{g(x)\} \quad \forall x \in \mathbf{R}^n$.

- If $T = \wedge$ then $r_{\min}, r_{\max} \in \mathfrak{F}_{TR}$ and $r_{\min} = R$, $r_{\max} = R \vee R_g$. The εTR–regression functions generated by $r_{\min}, r_{\max}$ are

$$f_{r_{\min}\varepsilon}(x) = \begin{cases} 0 & \text{if} \quad \varepsilon > \frac{s}{\sigma(x)}; \\ [g(x) - s_\varepsilon(x), g(x) + s_\varepsilon(x)] & \text{if} \quad \varepsilon \leq \frac{s}{\sigma(x)}, \end{cases}$$

$$f_{r_{\max}\varepsilon}(x) = f_{r_{\min}\varepsilon}(x) \cup \{g(x)\},$$

where $s_\varepsilon(x) = \sigma(x)\sqrt{\ln \frac{s}{\varepsilon\sigma(x)}}$.
From these follows that $f_R(x) = \{g(x)\} \quad \forall x \in \mathbf{R}^n$.

C. An other example is illustrated in Fig. 1.

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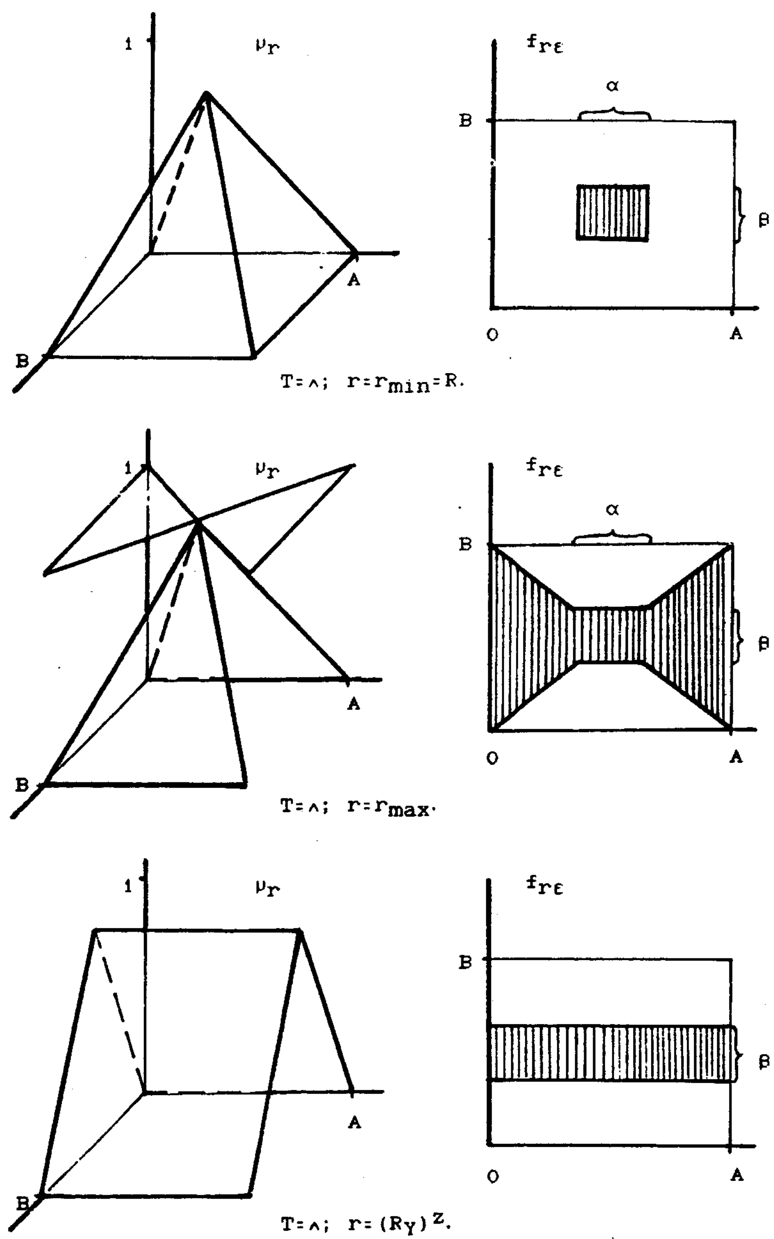

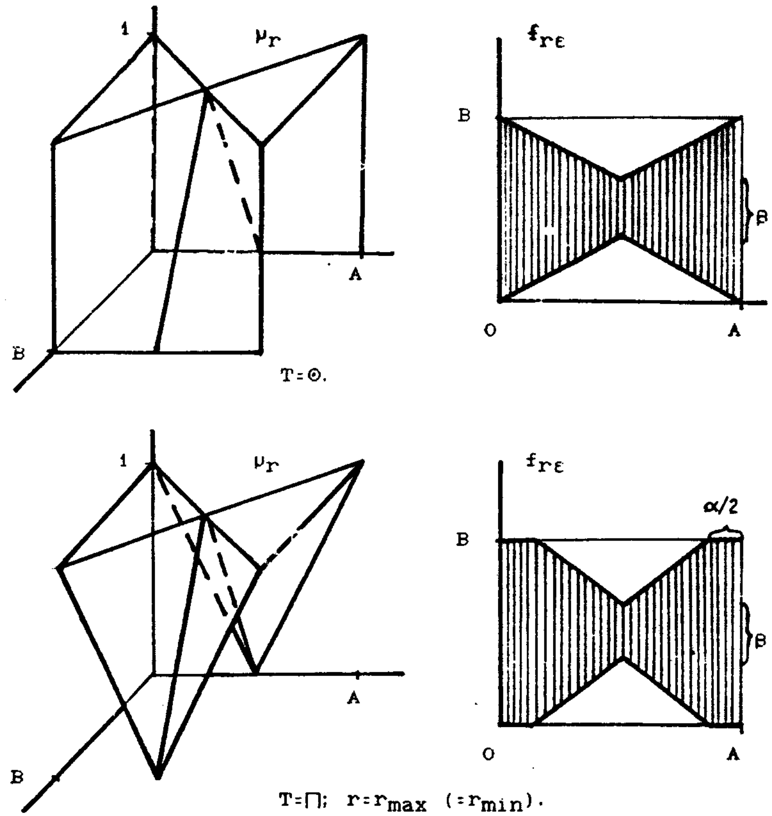

Figure 1. The functions μ_r ($r \in \mathcal{F}_{TR}(X \times Y)$ and $f_{r\varepsilon}$ for the fuzzy relation $R \in \mathcal{F}(X \times Y)$ defined by

$$\mu_R(x,y) = \min\{\max(0, 1 - |1 - \frac{2x}{A}|), \max(0, 1 - |1 - \frac{2y}{B}|)\}$$

and for t-norms $\wedge, \odot, \sqcap$ $(X = (0,A),\ Y = (0,B),\ \alpha = (1-A)\varepsilon,\ \beta = (1-B)\varepsilon)$.