EVERY BOUNDED SUBSET OF THE METRIC SPACE (E^n,D) IS RELATIVELY COMPACT

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1.Introduction

Denote E^n the set of strictly normal,fuzzy convex,upper semicontinuous,concave and compactly supported fuzzy numbers.Then (E^n,D) is a complete metric space with the metric D generated by the Hausdorff metric in $P_K(R^n)$ (see [1]),where $P_K(R^n)$ denotes the set of all nonempty, compact, convex subsets of R^n.We will prove that every bounded subset $K\subset E^n$ is relatively compact in (E^n,D).

2.Preliminaries

First we will give some definitions.

Definition2.1 Let (X,ρ) metric space.$K\subset X$ is bounded if there exists a number $M<+\infty$ such that $\rho(x,y)\leq M$ for each $x,y\in K$.

Definition2.2 Let X real linear space and let $K\subset X$.Then K is convex if $\alpha x+(1-\alpha)y\in K$ whenever $x,y\in K$ and $0\leq\alpha\leq 1$.

Definition2.3 Let (X,ρ) be metric space.Then $K\subset X$ is said to be relatively compact if every sequence in K has a subsequence that converges to a point in X.

We use the following notations (see in [1]).

Denote E^n the set of all strictly normal(i.e. there exists only one $t_o \in R^n$ such that $\tilde{x}(t_o) = 1$), fuzzy convex, upper semicontinuous, concave and compactly supported fuzzy numbers $\tilde{x}$ in R^n.

Let a function D be given as below.

$$D(\tilde{x},\tilde{y}) = \sup_{\alpha\in I} d([\tilde{x}]^\alpha, [\tilde{y}]^\alpha)$$

where I denotes the closed intervall [0,1],

d is the Hausdorff metric in $P_K(R^n)$,

$[\tilde{x}]^\alpha = \{t\in R \mid \tilde{x}(t)\geq\alpha\}$ for $0<\alpha\leq 1$,

$[\tilde{x}]^o = cl\{ t\in R \mid \tilde{x}(t)>0 \}$ the support of $\tilde{x}$, where cl(A) denotes the closure of the set A.

Then (E^n,D) is a complete metric space.

Definition2.3 The sequence $\{\tilde{x}_k\}\subset E^n$ converges to an $\tilde{x}_o\in E^n$ if

$$D(\tilde{x}_k,\tilde{x}_o) \to 0 \quad \text{as } k \to \infty$$

Definition2.4 The sequence $\{\tilde{x}_k\}\subset E^n$ converges to an $\tilde{x}_o\in E^n$ levelwise if

$$d([\tilde{x}_k]^\alpha, [\tilde{x}_o]^\alpha) \to 0 \quad \text{as } k \to \infty \quad \text{for all } 0<\alpha\leq 1$$

In the sequel we need the following theorems.

Theorem2.1 Convergence and levelwise convergence are equivalent in the metric space (E^n,D).

For proof see [2].

Theorem2.2 If $\{ A^\alpha \mid \alpha\in I \}$ is a family of subsets of R^n satisfying (i) - (iii), where

(i) $A^\alpha \in P_K(R^n)$ for all $\alpha\in I$

(ii) $A^\alpha \subset A^\beta$ for all $\alpha,\beta\in I, \alpha\geq\beta$

(iii) if $\{\alpha_k\}$ is a nondecreasing sequence converging to α

then

$$A^{\alpha} = \bigcap_{k\geq 1} A^{\alpha_k}$$

then there exists an $\tilde{x}\in E^n$ such that

$$[\tilde{x}]^{\alpha} = A^{\alpha} \quad \text{for } 0<\alpha\leq 1 \qquad \text{and}$$

$$[\tilde{x}]^{o} = \text{supp}(\tilde{x}) \subset A^{o}$$

For proof see[4].

We define addition and scalar multiplication in E^n according to Zadeh's extension principle.Then the metric D is translation invariant and positively homogenous(see[1]),consequently

$$\|\tilde{x}\| = D(\tilde{x},\hat{0}) \text{ forms a norm in } E^n.$$

3.The main theorem

<u>Theorem3.1</u> Every bounded subset K in (E^n,D) is relatively compact that is for every sequence $\{\tilde{x}_n\}\subset K$ there exists a subsequence $\{\tilde{x}_{\nu_k}\}$ of $\{\tilde{x}_k\}$ such that

$$\tilde{x}_{\nu_k} \text{ converges to an } \tilde{x}\in E^n.$$

<u>Proof</u>

Let us consider a sequence $\{\tilde{x}_k\}\subset K$.Then from the boundedness of K

2 $$D(\tilde{x}_k,0) \leq M <\infty \quad \text{for each } n\in N.$$

Consequently for all $k\in N$ and $\alpha\in I$ the nonempty compact,convex subsets $A_k^{\alpha} = [\tilde{x}_k]^{\alpha}$ are in a ball of R^n.Using the theorem Blashke (Theorem 14.5 in [3]) we have that for each $\alpha\in I$ there exists a subsequence ν_k and $A^\alpha \in P_K(R^n)$ such that

$$A^\alpha_{\nu_k} \to A^\alpha \quad \text{as } k \to \infty$$

Lemma3.1 Let us consider the family of subsets $\{ A^\alpha \mid \alpha \in I \}$ in Theorem3.1.Then the followings are valid.

(1) $A^\alpha \in P_K(R^n)$

(2) $A^\alpha = \bigcap_{k\geq 1} cl\{ \bigcup_{m\geq k} A^\alpha_{\nu_m} \} \qquad \alpha \in I$

(3) $A^\alpha \subset A^\beta \qquad$ for $\beta \leq \alpha$

(4) For $\alpha_k \uparrow \alpha > 0$

$$A^\alpha = \bigcap_{k\geq 1} A^{\alpha_k}$$

Proof of the lemma3.1

(1) and (2) can be found in [3].

(3) From the Theorem2.1 in [1] we know that

$$A^\alpha_{\nu_m} \subset A^\beta_{\nu_m} \quad \text{for } \beta \leq \alpha \quad \text{consequently}$$

$$\bigcup_{m\geq k} A^\alpha_{\nu_m} \subset \bigcup_{m\geq k} A^\beta_{\nu_m} \quad \text{for each } k \in N \qquad \text{so}$$

$$A^\alpha \subset A^\beta \ .$$

(4) Let us consider a sequence α_k such that $\alpha_k \uparrow \alpha > 0$.From the Theorem2.1 in[1] we have

$A_{\nu_m}^{\alpha_k} \subset A_{\nu_m}^{\alpha} \qquad$ for all $k \in N \qquad$ consequently

$$A^{\alpha_k} \subset A^{\alpha} \quad \text{for all } k\in N. \quad \text{That is}$$

$$\bigcap_{k\geq 1} A^{\alpha_k} \supset A^{\alpha}.$$

Conversely

$$A^{\alpha} = \bigcap_{k\geq 1} cl\{ \bigcup_{m\geq k} A_{\nu_m}{}^{\alpha} \} \quad \subset cl\{ \bigcup_{m\geq l} A_{\nu_m}{}^{\alpha} \} \quad \text{for all } l\in N.$$

$$\subset cl\{ \bigcup_{m\geq l} A_{\nu_m}{}^{\alpha_k} \} \quad \text{for all } k\in N \text{ and } l\in N.$$

$$\text{That is} \quad A^{\alpha} \subset \bigcap_{k\geq 1} A^{\alpha_k}.$$

The proof of the lemma is ended.
Now applying the Theorem2.2 and Theorem2.1 we can state that

there exists an $\tilde{x}_o\in E^n$ such that

$$[\tilde{x}_o]^{\alpha} = A^{\alpha} \qquad \alpha\in(0,1]$$

$$[\tilde{x}_o]^{o} \subset A^{o} \qquad \text{and}$$

$$\tilde{x}_{\nu_k} \to \tilde{x}_o \quad \text{in } (E^n,D).$$

The proof of the Theorem3.1 is ended.

<u>Corrolary3.1</u> If $K,F\subset E^n$ are nonempty sets such that K is compact and F is closed in (E^n,D) then there exist $\tilde{x}_o\in K$ and $\tilde{y}_o\in F$ such that

$$D(\tilde{x}_o,\tilde{y}_o) = D(K,F) = \inf_{\tilde{x}\in K,\tilde{y}\in F} D(\tilde{x},\tilde{y})$$

<u>Corrolary3.2</u> If $F \subset E^n$ closed nonempty set then there exists $\tilde{x}_o \in F$ such that

$$\|\tilde{x}_o\| = \inf_{\tilde{x} \in F} \|\tilde{x}\|$$

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