Fuzzy fixed point theorem of probability metric space

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This paper gives the concept of fuzzy mapping of probability metric space, and for I,III,IV class constriction mappings of probability metric space in [1] carry on fuzzy generalization, we obtain come fixed point theorems.

1. Preliminaries

$\mathscr{D}$ denotes the collection of all left continuous distribution function.

Definition1.1([1]). Let (E,F,Δ) be a Menger probability metric space(simply write MengerPM-space),where E is a abstract set, Δ is a t-norm, F is a mapping of $E\times E\to\mathscr{D}$ and satisfies the following condition(write $F(x,y)=F_{x,y}$, $F_{x,y}(t)$ denotes the value of distribution function $F_{x,y}$ at $t\in R=(-\infty,\infty)$):

(PM-1). $F_{x,y}(t)=1$ $\forall t>0$ if and only if $x=y$;

(PM-2). $F_{x,y}(0)=0$;

(PM-3). $F_{x,y}=F_{y,x}$;

(PM-4). $F_{x,z}(t_1+t_2)\geqslant\Delta(F_{x,y}(t_1),F_{y,z}(t_2))$ $\forall x,y,z\in E$, $t_1,t_2\geqslant 0$.

If (E,F,Δ) is a MengerPM-space with continuous t-norm Δ, then (E,F,Δ) is a Hausdorff space of the topology $\mathcal{T}$ induced by neighbourhood system $\{U_p(\varepsilon,\lambda)$: $p\in E$, $\varepsilon>0$, $\lambda>0\}$ ([3]),where $U_p(\varepsilon,\lambda)=\{x\in E$: $F_{x,y}(\varepsilon)>1-\lambda\}$. Let $\{x_n\}$ be any sequence in E. $\{x_n\}$ is called $\mathcal{T}$-concergence to $x^*\in E$, if for any $\varepsilon>0$, $\lambda>0$, there exists positive integer $N=N(\varepsilon,\lambda)$, as $n\geqslant N$, we have $F_{x_n,x^*}(\varepsilon)>1-\lambda$, write $x_n\xrightarrow{\mathcal{T}}x^*$; $\{x_n\}$ is called $\mathcal{T}$-Cauchy sequence in E, if for any $\varepsilon>0$, $\lambda>0$, there exists positive integer $N=N(\varepsilon,\lambda)$, as $n,m\geq N$, we have $F_{x_n,x_m}(\varepsilon)>1-\lambda$; (E,F,Δ) is called $\mathcal{T}$-complete, if each $\mathcal{T}$-Cauchy sequence in MengerPM-space is all $\mathcal{T}$-concergence to a point in E.

Theorem1.1([1.theorem9.2.9]). Let (E,F,Δ) be a MengerPM-space with continuous t-norm Δ, then $\{x_n\}\subset E$ $\mathcal{T}$-concergence to $x^*\in E$ if and only if for each $t\in R$, pointlike have

$$\lim_{n\to\infty} F_{x_n,x^*}(t)=H(t)=\begin{cases}1 & t>0\\ 0 & t\leq 0\end{cases}$$

Theorem1.2([1.corollary9.2.12]). Let (E,F,Δ) be a MengerPM-space with continuous t-norm Δ, let $y\in E$ be a fix point, and $x_n\xrightarrow{\mathcal{T}}x$, then $\lim_{n\to\infty}\inf F_{x_n,y}(t)\geq F_{x,y}(t)$ $\forall t\in R$. When t is continuous point of distribution function $F_{x,y}$, $\lim_{n\to\infty} F_{x_n,y}(t)=F_{x,y}(t)$.

Definition1.2([1]). Let (E,F,Δ) be a MengerPM-space, $A\subset E$ be a non-empty set. A is called probubility bounded set, if $\sup_{t>0}\inf_{x,y\in A}F_{x,y}(t)=\sup_{t>0}\sup_{s<t}\inf_{x,y\in A}F_{x,y}(s)=1$.

Let (E,F,Δ) be a MengerPM-space with continuous t-norm Δ, Ω denotes the collection of all non-empty $\mathcal{T}$-closed probability bounded sets. Now we define mapping $\tilde{F}$ on $\Omega\times\Omega$ as follows (write $\tilde{F}(A,B)=\tilde{F}_{A,B}$, $\tilde{F}_{A,B}(t)$ denotes the value of $\tilde{F}_{A,B}$ at t):

$$\tilde{F}_{A,B}(t)=\sup_{s<t}\left(\inf_{a\in A}\sup_{b\in B}F_{a,b}(s),\ \inf_{b\in B}\sup_{a\in A}F_{a,b}(s)\right)\quad \forall t>0$$

Now $(\Omega,\tilde{F},\Delta)$ is a MengerPM-space([2]). $(\Omega,\tilde{F},\Delta)$ is called MengerPM-space induced by (E,F,Δ), i.e. $\tilde{F}$ satisfies (PM-1) ——(PM-4).

For any $x\in E$, $A\in\Omega$, we define function

$$F_{x,A}(t)=\sup_{s<t}\sup_{y\in A}F_{x,y}(s)\quad \forall t>0.$$

Lemma1.1([2.proposition1]). Let (E,F,Δ) be a MengerPM-space with continuous t-norm Δ, $A\in\Omega$, then

(a). $F_{x,A}(t)=1$ $\forall t>0$ if and only if $x\in A$;

(b). $F_{x,A}(t_1+t_2)\geq\Delta(F_{x,y}(t_1),\ F_{y,A}(t_2))$ $\forall t_1,t_2\geq 0$, $y\in E$;

(c). For any $A,B\in\Omega$ and $x\in A$, we have

$$F_{x,B}(t)\geqslant\widetilde{F}_{A,B}(t)\quad\forall t\in R.$$

Lemma1.2. Let (E,F,Δ) be a MengerPM-space with continuous t-norm Δ, $\{x_n\}\subset E$, $A\in\Omega$, $x_n\xrightarrow{\mathcal{T}}x$, then

$$\liminf_{n\to\infty} F_{x_n,A}(t)\geqslant F_{x,A}(t)\quad\forall\ t>0.$$

Proof. By definition of $F_{x,A}(t)$ and theorem1.2, we have

$$\begin{aligned}\liminf_{n\to\infty} F_{x_n,A}(t)&=\liminf_{n\to\infty}\ \sup_{s<t}\ \sup_{y\in A} F_{x_n,y}(s)\quad\forall\ t>0\\ &\geqslant\sup_{s<t}\ \sup_{y\in A} F_{x,y}(s)\\ &=F_{x,A}(t)\end{aligned}$$

2. Main resulte

Let $\mathcal{F}(E)$ denotes the collection of all fuzzy sets in E, $\omega(E)$ denotes this conllection of fuzzy sets, for each $A\in\omega(E)$ set $\widetilde{A}\in\Omega$ is defined by the following equality

$$\widetilde{A}=\{x\in E:\ A(x)=\max_{u\in E}A(u)\}.$$

Definition2.1. Mapping $T: E\longrightarrow\omega(E)$ is called fuzzy mapping over E.

Thus, for each $x\in E$, $T(x)$ is a fuzzy set in $\omega(E)$, write $T(x)=T_x$, such that for any $y\in E$, $T_x(y)$ denotes the membership degree of y with respect to the fuzzy set T_x.

Definition2.2. The point x^* is called fixed point of fuzzy mapping T, if $T_{x^*}(x^*)\geqslant T_{x^*}(u)\ \ \forall u\in E$

By fuzzy mapping T over E, we define a Multi-valued mapmaings $\widetilde{T}: E\longrightarrow\Omega$:

$$\widetilde{T}(x)=\{y\in E:\ T_x(y)=\mathop{MAX}_{u\in E}\ T_x(u)\}\in\Omega.\qquad(2,1)$$

Theorem2.1. Let (E,F,Δ) be a $\mathcal{T}$-complete MengerPM-space with continuous t-norm $\Delta(\Delta(a,b)\geqslant\max\{a+b-1,\ 0\}\ \ \forall a,b\in[0,1])$, T is a fuzzy mapping over E, $\widetilde{T}$ is the Multi-valued mappings is defined by T according to (2,1), for any $x,y\in E$, $k\in(0,1)$,

$$\tilde{F}_{\tilde{T}(x),\tilde{T}(y)}(t) \geq F_{x,y}\left(\frac{t}{k}\right) \qquad \forall\, t \geq 0, \qquad (2,2)$$

then T there exists fixed point $x^* \in E$.

Proof. $\forall x_0 \in E$, by (PM-4), we taking $x_1 \in \tilde{T}(x_0)$, $x_2 \in \tilde{T}(x_1)$, for any $r>1$ such that

$$F_{x_1,x_2}(t) \geq \Delta\left(F_{x_1,\tilde{T}(x_0)}\left(\left(1-\frac{1}{r}\right)t\right), F_{\tilde{T}(x_0),x_2}\left(\frac{t}{r}\right)\right) \quad \forall\, t \geq 0$$

$$\geq F_{\tilde{T}(x_0),x_2}\left(\frac{t}{r}\right) \qquad \text{by Lemma1.1(a)}$$

letting $r \longrightarrow 1$, by left continuity of distribution function, we obtain

$$F_{x_1,x_2}(t) \geq F_{\tilde{T}(x_0),x_2}(t) \qquad \forall\, t \geq 0$$

$$\geq \tilde{F}_{\tilde{T}(x_0),\tilde{T}(x_1)}(t) \qquad \text{by Lemma1.1(c)}$$

$$\geq F_{x_0,x_1}\left(\frac{t}{k}\right) \qquad \text{by (2,2)}$$

Repeatedly above procedure we obtain sequence $\{x_n\} \subset E$, such that

$$x_n \in \tilde{T}(x_{n-1}) \in \Omega \qquad n=1,2,\cdots$$

$$F_{x_n,x_{n+1}}(t) \geq \tilde{F}_{\tilde{T}(x_{n-1}),\tilde{T}(x_n)}(t) \quad \forall\, t \geq 0,\ n=1,2,\cdots \qquad (2,3)$$

Next we prove that $\{x_n\}$ is a $\mathcal{J}$-Cauchy sequence. In fact, any given positive integers K, m, we have

$$F_{x_K,x_{K+m}}(t) \geq \tilde{F}_{\tilde{T}(x_{K-1}),\tilde{T}(x_{K+m-1})}(t) \qquad \text{by (2,3)}$$

$$\geq F_{x_{K-1},x_{K+m-1}}\left(\frac{t}{k}\right) \qquad \text{by (2,2)}$$

$$\geq \tilde{F}_{\tilde{T}(x_{K-2}),\tilde{T}(x_{K+m-2})}\left(\frac{t}{k^2}\right)$$

$$\geq F_{x_{K-2},x_{K+m-2}}\left(\frac{t}{k^2}\right)$$

$$\geq \cdots$$

$$\geq F_{x_0,x_m}\left(\frac{t}{k^K}\right)$$

since $k \in (0,1)$, as $K \longrightarrow \infty$, $\frac{1}{k^K} \longrightarrow \infty$, hence

$$\lim_{K\to\infty} F_{x_K,x_{K+m}}(t) \geq H(t) = \begin{cases} 1 & t>0 \\ 0 & t=0 \end{cases}$$

it follows that the $\{x_n\}_{n=1}$ is a $\mathcal{J}$-Cauchy sequence in E. Since E is $\mathcal{J}$-complete, thus there exists $x^*\in E$ such that $x_n \xrightarrow{\mathcal{J}} x^*$.

Now we prove that x^* is a fixed point of T. Suffice it to prove $x^*\in\tilde{T}(x^*)$. In fact, any given positive integer n, we have

$$F_{x_n,\tilde{T}(x^*)}(t)\geq\tilde{F}_{\tilde{T}(x_{n-1}),\tilde{T}(x^*)}(t) \quad \text{by Lemma1.1(c)}$$
$$\geq F_{x_{n-1},x^*}(\tfrac{t}{K}) \quad \text{by (2,2)}$$

Letting $n\longrightarrow\infty$ we have

$$\lim_{n\to\infty} F_{x_n,\tilde{T}(x^*)}(t)=1 \quad \forall t>0 \quad \text{by theorem1.1} \qquad (2,4)$$

Moreover

$$F_{x^*,\tilde{T}(x^*)}(t)\geq\Delta(F_{x^*,x_n}(\tfrac{t}{2}),\ F_{x_n,\tilde{T}(x^*)}(\tfrac{t}{2})) \quad \forall t\geq 0$$

as $n\longrightarrow\infty$, by (2,4),

$$F_{x^*,\tilde{T}(x^*)}(t)=1 \quad \forall\ t>0.$$

By Lemma1.1(a), $x^*\in\tilde{T}(x^*)=\{y\in E:\ T_{x^*}(x^*)=\max_{u\in E} T_{x^*}(u)\}$. Thus

$$T_{x^*}(x^*)\geq T_{x^*}(u) \quad \forall\ u\in E\ .$$

By definition of fixed point of fuzzy mapping T, the x^* is a fixed point of fuzzy mapping T.

We prove that fixed point is only. Let we choose two sequence $\{x_n\}\subset E$, $\{y_n\}\subset E$ by this way, $x_n\xrightarrow{\mathcal{J}}x^*\in E$, $y_n\xrightarrow{\mathcal{J}}y^*\in E$, $x^*\in\tilde{T}(x^*)$, $y^*\in\tilde{T}(y^*)$.

$$F_{x^*,y^*}(t)\geq\tilde{F}_{\tilde{T}(x^*),\tilde{T}(y^*)}(t)\geq F_{x^*,y^*}(\tfrac{t}{K})\geq\cdots\geq F_{x^*,y^*}(\tfrac{t}{K^n})$$

letting $n\longrightarrow\infty$, we have

$$F_{x^*,y^*}(t)=1 \quad \forall\ t>0$$

thus $\quad x^*=y^*$.

Theorem2.2. Let (E,F,Δ) be a $\mathcal{J}$-complete MengerPM-space with continuous t-norm Δ, $(\Delta(a,b)\geq\max\langle a+b-1,\ 0\rangle\ \forall a,b\in[0,1])$ Let $\{T^i\}_{i=1}$ be the fuzzy mappings sequence over E, $\langle\tilde{T}_i\rangle_{i=1}$ is the Multi-valued mappings sequence defined by $\{T^i\}_{i=1}$

according to (2,1) respectively. Let there exists constant $k>1$, such that for any positive integers i,j, $i\neq j$ and any x $y\in E$ have

$$\widetilde{F}_{\widetilde{T}_i(x),\widetilde{T}_j(y)}(t)\geq\min\{F_{x,y}(kt),F_{x,\widetilde{T}_i(x)}(kt),F_{y,\widetilde{T}_j(y)}(kt)\} \quad \forall\, t\geq 0 \qquad (2,5)$$

then there exists $x^*\in E$ is a common fixed point of $\{T^i\}_{i=1}$.

Proof. $\forall x_o\in E$, we taking $x_1\in\widetilde{T}_1(x_o)\in\Omega$, $x_2\in\widetilde{T}_2(x_1)\in\Omega$, for any $r>1$, we have

$$F_{x_1,x_2}(t)\geq\Delta(F_{x_1,\widetilde{T}_1(x_o)}((1-\frac{1}{r})t),\ F_{\widetilde{T}_1(x_o),x_2}(\frac{t}{r})\} \quad \forall\, t\geq 0$$

$$\geq F_{\widetilde{T}_1(x_o),x_2}(\frac{t}{r}) \qquad \text{by Lemma1.1(a)}$$

letting $r\longrightarrow 1$, by left continuity of distribution function, we obtain

$$F_{x_1,x_2}(t)\geq F_{\widetilde{T}_1(x_o),x_2}(t) \quad \forall\, t\geq 0$$

$$\geq\widetilde{F}_{\widetilde{T}_1(x_o),\widetilde{T}_2(x_1)}(t) \qquad \text{by Lemma1.1(c)}$$

$$\geq\min\{F_{x_o,x_1}(kt),F_{x_o,\widetilde{T}_1(x_o)}(kt),F_{x_1,\widetilde{T}_2(x_1)}(kt)\}$$

$$\text{by } (2,5)$$

For any $r>1$,

$$F_{x_o,\widetilde{T}_1(x_o)}(kt)\geq\Delta(F_{x_o,x_1}(\frac{kt}{r}),F_{x_1,\widetilde{T}_1(x_o)}((1-\frac{1}{r})kt))$$

$$\text{by Lemma1.1(b)}$$

$$\geq F_{x_o,x_1}(\frac{kt}{r}) \qquad \text{by Lemma1.1(a)}$$

letting $r\rightarrow 1$, we obtain

$$F_{x_o,\widetilde{T}_1(x_o)}(kt)\geq F_{x_o,x_1}(kt) \quad \forall\, t\geq 0$$

by left continuity of distribution function.

Using the same method, we can obtain

$$F_{x_1,\widetilde{T}_2(x_1)}(kt)\geq F_{x_1,x_2}(kt)$$

thus $$F_{x_1,x_2}(t)\geq\min\{F_{x_o,x_1}(kt),\ F_{x_1,x_2}(kt)\}$$

since $k>1$ hence

$$F_{x_1,x_2}(t) \geq F_{x_o,x_1}(kt) \qquad \forall\, t \geq 0 \qquad (2,6)$$

Repeatealy above procedure, we obtain sequence $\{x_n\}_{n=1}$ such that

$$x_n \in \tilde{T}_n(x_{n-1}) \in \Omega \qquad n=1,2,\cdots$$

$$F_{x_n,x_{n+1}}(t) \geq \tilde{F}_{\tilde{T}_n(x_{n-1}),\tilde{T}_{n+1}(x_n)}(t) \geq F_{x_o,x_1}(k^n t) \forall t \geq 0 \qquad (2,7)$$

Next we prove that $\{x_n\}_{n=1} \subset E$ is a $\mathcal{J}$-Cauchy sequence. In fact, any given positive integers K,m, we have

$$F_{x_K,x_{K+m}}(t) \geq \tilde{F}_{\tilde{T}_K(x_{K-1}),\tilde{T}_{K+m}(x_{K+m-1})}(t) \quad \text{by } (2,7)$$

$$\geq \min\{F_{x_{K-1},x_{K+m-1}}(kt), F_{x_{K-1},\tilde{T}_K(x_{K-1})}(kt), F_{x_{K+m-!},\tilde{T}_{K+m}(X_{K+m-1})}(kt)\} \quad \text{by } (2,5)$$

$$\geq F_{x_{K-1},x_{K+m-1}}(kt) \quad \text{by proof of } (2,6)$$

$$\geq \cdots$$

$$\geq F_{x_o,x_m}(k^K t) \ \forall\, t \geq 0 \quad \text{by } (2,7)$$

since $k>1$, letting $K \longrightarrow \infty$, we have

$$\lim_{K\to\infty} F_{x_K,x_{K+m}}(t) \geq 1 \qquad \forall\, t>0$$

it follows that the $\{x_n\}_{n=1}$ is a $\mathcal{J}$-Cauchy sequence. Since E is $\mathcal{J}$-complete, hence there exists $x^* \in E$ such that $x_n \xrightarrow{\mathcal{J}} x^*$.

Now we prove that $x^* \in \bigcap_{i=1}^{\infty} \tilde{T}_i(x^*)$. In fact, any given positive integers i,n, $i \neq n$, we have

$$F_{x_n,\tilde{T}_i(x^*)}(t) \geq \tilde{F}_{\tilde{T}_n(x_{n-1}),\tilde{T}_i(x^*)}(t) \ \forall t \geq 0 \text{ by lemma1.1(c)}$$

$$\geq \min\{F_{x_{n-1},x^*}(kt), F_{x_{n-1},x_n}(kt), F_{x^*,\tilde{T}_i(x^*)}(kt)\} \qquad \text{by proof of } (2,6)$$

thus $$\lim_{n\to\infty} F_{x_n,\tilde{T}_i(x^*)}(t) \geq \lim_{n\to\infty} F_{x^*,\tilde{T}_i(x^*)}(kt)$$

moreover for any $r>1$

$$F_{x^*,\tilde{T}_i(x^*)}(kt)\geqslant\Delta(F_{x^*,x_n}((1-\frac{1}{r})kt),F_{x_n,\tilde{T}_i(x^*)}(\frac{kt}{r}))$$

$$\lim_{n\to\infty} F_{x^*,\tilde{T}_i(x^*)}(kt)\geqslant\lim_{n\to\infty} F_{x_n,\tilde{T}_i(x^*)}(\frac{kt}{r})$$

letting $r\to 1$, we have

$$\lim_{n\to\infty} F_{x_n,\tilde{T}_i(x^*)}(t)\geqslant\lim_{n\to\infty} F_{x^*,\tilde{T}_i(x^*)}(kt)\geqslant\lim_{n\to\infty} F_{x_n,\tilde{T}_i(x^*)}(kt)\geqslant\lim_{n\to\infty} F_{x^*,\tilde{T}_i(x^*)}(k^2t)\geqslant\cdots\geqslant\lim_{n\to\infty} F_{x^*,\tilde{T}_i(x^*)}(k^nt)=1\quad\forall t>0$$

$$F_{x^*,\tilde{T}_i(x^*)}(t)\geqslant\Delta(F_{x^*,x_n}(\frac{t}{2}),F_{x_n,\tilde{T}_i(x^*)}(\frac{t}{2}))$$

letting $n\to\infty$, we obtain

$$F_{x^*,\tilde{T}_i(x^*)}(t)=1\qquad\forall\ t>0$$

thus $\quad x^*\in\tilde{T}_i(x^*)\quad i=1,2,\cdots$, i.e. $x^*\in\bigcap_{i=1}\tilde{T}_i(x^*)$

So the x^* is a common fixed point of fuzzy mappings sequence $\{T^i\}_{i=1}$.

Theorem2.3. Let (E,F,Δ) be a $\mathcal{T}$-complete MengerPM-space with continuous t-norm $\Delta(\Delta(a,b)\geqslant\max\{a+b-1,\ 0\}\ a,b\in[0,1])$, let $\{T^i\}_{i=1}$ be the fuzzy mappings sequence over E, $\{\tilde{T}_i\}_{i=1}$ is the Multi-valued mappings sequence defined by $\{T^i\}_{i=1}$ according to (2,1) respectively. For any $x,y\in E$, any positive integers i,j, $i\neq j$ have

$$\tilde{F}_{\tilde{T}_i(x),\tilde{T}_j(y)}(t)\geqslant F_{x,y}(\phi(t))\qquad\forall\ t\geqslant 0\qquad(2,8)$$

where ϕ is a function of satisfying the following condition (ϕ_1):

(ϕ_1). ϕ: $[0,\infty)\longrightarrow[0,\infty)$, $\phi(0)=0$, $\phi(t)$ is strict increase with respect to t and for all $t>0$ $\lim\phi^n(t)=\infty$, where $\phi^n(t)$ is n-th iteration of $\phi(t)$, then $\{T^i\}_{i=1}$ there exists common fixed point.

Proof. Follow suit theorem2.1, $\forall\, x_0\in E$ we choose sequence $\{x_n\}_{n=1}\subset E$ such that

$$x_n\in\tilde{T}_n(x_{n-1})\in\Omega,\ n=1,2,\cdots$$

$$F_{x_n,x_{n+1}}(t)\geq F_{\widetilde{T}_n(x_{n-1}),x_{n+1}}(t)\geq \widetilde{F}_{\widetilde{T}_n(x_{n-1}),\widetilde{T}_{n+1}(x_n)}(t)$$
$$n=1,2,\cdots \qquad (2,9)$$

We prove that $\{x_n\}_{n=1}\subset E$ is a $\mathcal{J}$-Cauchy sequence. In fact, any given positive integers K,m, we have

$$F_{x_K,x_{K+m}}(t)\geq \widetilde{F}_{\widetilde{T}_K(x_{K-1}),\widetilde{T}_{K+m}(x_{K+m-1})}(t) \quad \text{by } (2,9)$$
$$\geq F_{x_{K-1},x_{K+m-1}}(\phi(t)) \quad \forall\, t\geq 0 \quad \text{by } (2,8)$$
$$\geq \widetilde{F}_{\widetilde{T}_{K-1}(x_{K-2}),\widetilde{T}_{K+m-1}(x_{K+m-2})}(\phi(t))$$
$$\geq F_{x_{K-2},x_{K+m-2}}(\phi^2(t))$$
$$\geq \cdots$$
$$\geq F_{x_0,x_m}(\phi^K(t))$$

by (ϕ_1), $\phi(0)=0$, $\forall t>0$, $\phi(t)$ is strict increase , thus $\phi(t)>0$ and $\lim\limits_{n\to\infty}\phi^n(t)=\infty$, letting $K\to\infty$ we obtain

$$\lim_{K\to\infty} F_{x_K,x_{K+m}}(t)\geq 1 \quad \forall\, t>0$$

it follows that the $\{x_n\}_{n=1}$ is a $\mathcal{J}$-Cauchy sequence in E. moreover E is $\mathcal{J}$-complete, there exists $x^*\in E$, such that $x_m\xrightarrow{\mathcal{J}}x^*$.

Now we prove that x^* is a common fixed point of fuzzy mappings sequence $\{T^i\}_{i=1}$. i.e. $x^*\in\bigcap\limits_{i=1}\widetilde{T}_i(x^*)$. In fact, any given positive integers i,n, $i\neq n$, we have

$$F_{x_n,\widetilde{T}_i(x^*)}(t)\geq \widetilde{F}_{\widetilde{T}_n(x_{n-1}),\widetilde{T}_i(x^*)}(t) \quad \forall\, t\geq 0 \text{ by lemma } 1.1(c)$$
$$\geq F_{x_{n-1},x^*}(\phi(t)) \qquad \text{by } (2,8)$$

letting $n\to\infty$, by (ϕ_1), $\forall t>0$ have

$$\lim_{n\to\infty} F_{x_n,\widetilde{T}_i(x^*)}(t)=1 \quad \forall\, t>0 \qquad (2,10)$$

other $$F_{x^*,\widetilde{T}_i(x^*)}(t)\geq\Delta\left(F_{x^*,x_n}\left(\tfrac{t}{2}\right),F_{x_n,\widetilde{T}_i(x^*)}\left(\tfrac{t}{2}\right)\right)$$

by lemma1.1(b)

letting $n \to \infty$ by theorem1.1 and (2,10)

$$F_{x^*, \widetilde{T}_i(x^*)}(t)=1 \qquad \forall t>0 \quad i=1,2,\cdots$$

it follows that the $x^* \in \bigcap_{i=1}^{\infty} \widetilde{T}_i(x^*)$.

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