FIXED DEGREE THEOREMS OF FUZZY MAPPING

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This paper obtains some new fixed degree theorems of fuzzy mapping by concept of fixed degree of generalized fuzzy mappings. The results given in this paper improve and extend some results in [1] by Chang Shihsen.

1. Preliminaries

Throughout this paper (X,d) denotes a complete metric space; $H(\cdot,\cdot)$, the Hausdorff metric induced by metric d; C(X), the collection of all non-empty compact subsets of X; $\mathscr{F}(X)$, the collection of all fuzzy sets in X. Let $A\in\mathscr{F}(X)$, $\alpha\in(0,1]$. We write

$$\text{supp } A=\{x\in X:\ A(x)>0\};$$

$$(A)_\alpha=\{x\in X:\ A(x)=\alpha\};$$

$$\langle A\rangle_\alpha=\{x\in X:\ A(x)\geqslant\alpha\};$$

$$\tilde{A}=\{\xi^x_\lambda:\ x\in X,\ A(x)=\lambda\in(0,1]\},$$

where ξ^x_λ is a fuzzy point which takes x as supporting point, λ as value.

Definition 1. Let $A\in\mathscr{F}(X)$, $F:\tilde{A}\longrightarrow\mathscr{F}(X)$ be a mapping, which is called a fuzzy mapping over A, if for each $\xi^x_\lambda\in\tilde{A}$, we have $F(\xi^x_\lambda)\subset A$. We write $F(\xi^x_\lambda)=F_{\xi^x_\lambda}$.

Clearly, if A is an obvious set, then the fuzzy mapping defined above is considered in [1]. The set-valued mapping T: $X \longrightarrow 2^X$ can be taken as a special case of above mentioned fuzzy mapping.

Definition 2. Let $A \in \mathscr{F}(X)$, F be a fuzzy mapping over A, $\xi_\lambda^x \in \tilde{A}$. If $F_{\xi_\lambda^x}(x)=\alpha$, the $\frac{\alpha}{\lambda}$ is called fixed degree of ξ_λ^x for fuzzy mapping F, we write $D_{fix}(\xi_\lambda^x, F)=\frac{\alpha}{\lambda}$.

Specifically if $D_{fix}(\xi_\lambda^x, F)=1$, i.e. $F_{\xi_\lambda^x}(x)=\lambda$, then ξ_λ^x is called fixed point of F. If $F_{\xi_\lambda^x}(x)=\max_{u \in X} F_{\xi_\lambda^x}(u)$, then we say that F obtains maximal fixed degree at fuzzy point ξ_λ^x.

Let $A \in \mathscr{F}(X)$, F be a fuzzy mapping over A, if for any $x \in$ supp A, there exists a corresponding $\alpha(x) \in (0,1]$ such that $\{y \in X: F_{\xi_{A(x)}^x}(y)=\alpha(x)\} \in C(X)$, then we can define a set-valued mapping $\hat{F}$: supp $A \longrightarrow C(X)$ as follows:

$$\hat{F}(x)=\{y \in X: F_{\xi_{A(y)}^y}(y)=\alpha(x)\} \text{ for } \forall x \in \text{supp } A. \quad (1.1)$$

Cleary, for any $x \in$ supp A, we have $\hat{F}(x) \subset$ supp A, thus for any $y \in \hat{F}(x)$ we have $\xi_{A(x)}^x \in \tilde{A}$. From the definition we can immediately obtain the following result.

Lemma 1. Let $A \in \mathscr{F}(X)$, F be a fuzzy mapping over A, $\hat{F}$ be the set-valued mapping defined by F according to (1.1). Then fixed degree of $\xi_{A(x)}^x \in \tilde{A}$ with respect to F is equal to $\frac{\alpha(x)}{A(x)}$ if and only if x is fixed point of the set-valued mapping $\hat{F}$, i.e. $x \in \hat{F}(x)=\{y \in X: F_{\xi_{A(x)}^x}(y)=\alpha(x)\}$.

2. Main results

Theorem 1. Let $A\in\mathscr{F}(X)$, $\langle A\rangle_r\in \mathbb{C}(X)$, $0<r<1$, F, G be two fuzzy mappings over A. If for ang $x, y\in$ supp A there are corresponding $\alpha(x)$, $\beta(y)\in[r, 1]$ such that $(F_{\xi^x_{A(x)}})_{\alpha(x)}$, $(G_{\xi^y_{A(y)}})_{\beta(y)}\in C(X)$, and

$$H((F_{\xi^x_{A(x)}})_{\alpha(x)}, (G_{\xi^y_{A(y)}})_{\beta(y)})\leq \Phi(d(x, y), d(x, (F_{\xi^x_{A(x)}})_{\alpha(x)}), d(y, (G_{\xi^y_{A(y)}})_{\beta(y)}), d(x, G_{\xi^y_{A(y)}})_{\beta(y)}), d(y, (F_{\xi^x_{A(x)}})_{\alpha(x)})), \tag{2.1}$$

where the function Φ: $[0,\infty)^5\longrightarrow[0,\infty)$ satisfies the following conditions:

(Φ_1) Φ is strictly increasing for each variable, and Φ is upper semi-continuous; (2.2)

(Φ_2) $\Phi(t,t,t,at,bt)\leq\varphi(t)$, $\forall\, t\geq 0$ $a,b=0,1,2$; $a+b=2$; (2.3)

where φ: $[0,\infty)\longrightarrow[0,\infty)$ satisfies the conditions: $\varphi(t)<t$, $\forall\, t>0$; $\varphi(0)=0$. (2.4)

Then there exists a fuzzy point $\xi^{x^*}_{A(x^*)}\in\tilde{A}$ such that the common fixed degree of $\xi^{x^*}_{A(x^*)}$ for F and G is equal to

$$\min\left\{\frac{\alpha(x^*)}{A(x^*)}, \frac{\beta(x^*)}{A(x^*)}\right\}.$$

Proof. Let $\hat{F}$, $\hat{G}$: supp $A\longrightarrow C(X)$ be two set-valued mappings defined by F and G according to (1.1) respectively. By using Lemma 1, it is sufficient to prove that there exists $x^*\in$ supp A such that $x^*\in(\hat{F}(x^*)\cap\hat{G}(x^*))$.

Denote

$$\gamma_F=\inf\{ d(y, \hat{F}(y)),\ \ y\in \text{supp } A\};$$

$$r_G = \inf\{\, d(y, \hat{G}(y)), \quad y \in \operatorname{supp} A \,\}.$$

Let $\{y_n\} \subset \operatorname{supp} A$ be a sequence such that $d(y_n, \hat{F}(y_n)) \longrightarrow r_F$. Because of compactness of $\hat{F}(y_n)$, there exists $x_n \in \hat{F}(y_n)$ such that

$$d(y_n, x_n) = d(y_n, \hat{F}(y_n)), \quad n=1,2,\ldots. \tag{2.5}$$

Since $x_n \in \hat{F}(y_n)$, it follows from definitions of F and $\hat{F}$ that

$$A(x_n) \geqslant R_{E x_n \supseteq A(y_n)}(x_n) = \alpha(y_n) \geqslant r. \quad n=1,2,\ldots.$$

hence $\{x_n\} \subset \langle A \rangle_r$. Because of the compactness of $\langle A \rangle_r$, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges to $x^* \in \langle A \rangle_r \subset \operatorname{supp} A$.

We prove that the following hold:

$$x^* \in (\hat{F}(x^*) \cap \hat{G}(x^*)).$$

In fact,

$$d(x^*, G(x^*)) \leqslant d(x^*, x_{n_i}) + d(x_{n_i}, \hat{G}(x^*))$$
$$\leqslant d(x^*, x_{n_i}) + H(\hat{F}(y_{n_i}), \hat{G}(x^*)). \tag{2.6}$$

However,

$$H(\hat{F}(y_{n_i}), \hat{G}(x^*)) \leqslant \Phi(d(y_{n_i}, x^*), d(y_{n_i}, \hat{F}(y_{n_i})), d(x^*, \hat{G}(x^*)),$$
$$d(y_{n_i}, \hat{G}(x^*)), d(x^*, \hat{F}(y_{n_i})))$$
$$\leqslant \Phi(d(y_{n_i}, x_{n_i}) + d(x_{n_i}, x^*), d(y_{n_i}, x_{n_i}) + d(x_{n_i}, \hat{F}(y_{n_i})),$$
$$d(x^*, \hat{G}(x^*)), d(y_{n_i}, x_{n_i}) + d(x_{n_i}, x^*) +$$
$$d(x^*, \hat{G}(x^*)), d(x^*, x_{n_i}) + d(x_{n_i}, \hat{F}(y_{n_i})))$$

Substituting the above expression into (2.6), and leting $n_i \longrightarrow \infty$, we have

$$d(x^*, \hat{G}(x^*)) \leqslant \Phi(r_F, r_F, d(x^*, \hat{G}(x^*)), r_F + d(x^*, \hat{G}(x^*)),$$

0) (2.7)

If $\gamma_F < d(x^*, \hat{G}(x^*))$, it follows from the strictly increasing property of Φ that

$$d(x^*, \hat{G}(x^*)) \leqslant \Phi(d(x^*, \hat{G}(x^*)), d(x^*, \hat{G}(x^*)), d(x^*, \hat{G}(x^*)), 2d(x^*, \hat{G}(x^*)), 0)$$

$$\leqslant \varphi(d(x^*, \hat{G}(x^*)) < d(x^*, \hat{G}(x^*)).$$

This is a contradiction. By this contradiction we have $d(x^*, \hat{G}(x^*)) \leqslant \gamma_F$. Hence $\gamma_G \leqslant \gamma_F$.

By the symmetric property of F and G, we can similarly prove $\gamma_F \leqslant \gamma_G$. Hence we have $\gamma_F = \gamma_G = d(x^*, \hat{G}(x^*))$.

Now we prove $\gamma_F = 0$. Suppose this is not the case, $\gamma_F = d(x^*, \hat{G}(x^*) > 0$. Since $\hat{G}(x^*)$ is nonempty and compact, there exists $z_0 \in \hat{G}(x^*)$ such that $d(x^*, z_0) = \gamma_F > 0$. Therefore we have

$$d(z_0, \hat{F}(z_0) \leqslant H(\hat{G}(x^*), \hat{F}(z_0))$$

$$\leqslant \Phi(d(x^*, z_0), d(x^*, z_0)+d(z_0, \hat{G}(x^*)), d(z_0, \hat{F}(z_0)), d(x^*, z_0)+d(z_0, \hat{F}(z_0)), 0) \quad (2.8)$$

From (2.8) we can prove $d(z_0, \hat{F}(z_0)) < d(x^*, z_0)$, hence we have

$$\gamma_F \leqslant d(z_0, \hat{F}(z_0)) < \gamma_F.$$

This is a contradiction. From this contradiction it follows that

$$d(z_0, \hat{F}(z_0)) = \gamma_F = \gamma_G = d(x^*, \hat{G}(x^*)) = d(x^*, z_0) = 0$$

and $x^* \in (\hat{F}(x^*) \cap \hat{G}(x^*))$.

Therefore we have

$$D_{fix}\ (\xi^{x^*}_{A(x^*)},\ F) = \frac{\alpha(x^*)}{A(x^*)}$$

and

$$D_{fix}\ (\xi^{x^*}_{A(x^*)},\ G) = \frac{\beta(x^*)}{A(x^*)}\ .$$

Therefore the common fixed degree of $\xi^{x^*}_{A(x^*)}$ for F and G is equal to

$$\min\left\{\frac{\alpha(x^*)}{A(x^*)}\ .\ \frac{\beta(x^*)}{A(x^*)}\right\}.$$

By using similar way of Theorem 1 we can prove the following result.

Theorem 2. Let $A \in \mathscr{F}(X)$, $\langle A \rangle_r \in C(X)$, $0 < r < 1$, $\{F_i\}_{i=1}^{\infty}$ be a sequence of fuzzy mappings over A. Suppose that for any x, $y \in$ supp A, and any positive integers i, j, $i \neq j$ there are corresponding $\alpha_i(x) \in [r,\ 1]$ such that

$$(F_i\,\xi^x_{A(x)}\,)_{\alpha_i(x)} \in C(X) \tag{2.9}$$

and

$$\begin{aligned} H((F_i\,\xi^x_{A(x)}\,)_{\alpha_i(x)}\ ,\ (F_j\,\xi^y_{A(y)}\,)_{\alpha_j(y)}\ ) \\ \leqslant \Phi(d(x,\ y),\ d(x,\ (F_i\,\xi^x_{A(x)}\,)_{\alpha_i(x)}\ ),\ d(y,\ (F_j\,\xi^y_{A(y)}\,)_{\alpha_j(y)}), \\ d(x,\ (F_j\,\xi^y_{A(y)}\,)_{\alpha_j(y)}),\ d(y,\ (F_i\,\xi^x_{A(x)}\,)_{\alpha_i(x)}\ )) \end{aligned} \tag{2.10}$$

where the function Φ satisfies (Φ_1) and (Φ_2), φ satisfies (2.4). Then there exists a fuzzy point $\xi^{x^*}_{A(x^*)} \in \tilde{A}$ such that the common fixed degree of $\xi^{x^*}_{A(x^*)}$ for $\{F_i\}_{i=1}^{\infty}$ is equal to

$$\inf\left\{\frac{\alpha_1(x^*)}{A(x^*)}\right\}.$$

References

[1] Chang Shihsen, Fixed Point theorems for fuzzy mappings, Fuzzy sete and systems 17 (1985) 181-187.

[2] Fang Jinxuan, Fixed degree of fuzzy mappings, Science Bulletin 30 (1985) 8:635 (in Chinese).

[3] S.Y.Liu Fixed degree theorems of generalized fuzzy mapping, BUSEFAL 32 (1987) 85-91.