GREY PROBABILITY

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ABSTRAACT: The classical theory of Probability was established on the base of the theory of Cantor's set, and Fuzzy's is establised on that of Fuzzy'sset. Similarly, we can have the grey probability based on the theory of grey set. This paper aims at establishing the conception of grey probability and discussing its basic properties so as to initiate the development of grey probability.

I. Basic Space and Grey Events

DEFINITION 1: Let universe of discourse U be a Cantor's set, which is, in order to be identical with classical mathematics, also called the basic space.

DEFINITION 2: Let $\underset{\otimes}{D}$ be a grey subset of U, then $\underset{\otimes}{D}$ is a grey event, and U is called an inevitable event, empty set is called an impossible event, and the event composed of simple elements in the basic space is called the basic event.

DEFINITION 3: $\underset{\otimes}{A}\cup\underset{\otimes}{B}$, and $\underset{\otimes}{A}\cap\underset{\otimes}{B}$ are respectively called the sum event and the product event of $\underset{\otimes}{A}$ and $\underset{\otimes}{B}$, $\underset{\otimes}{A}\cap\underset{\otimes}{B}^c$ is called the difference event of $\underset{\otimes}{A}$ and $\underset{\otimes}{B}$, and $\underset{\otimes}{A}^c$ is called the inverse grey event.

DEFINITION 4: If $\underset{\otimes}{A}\subseteq\underset{\otimes}{B}$, $\underset{\otimes}{A}$ is the subevent of $\underset{\otimes}{B}$.

DEFINITION 5: If $\underset{\otimes}{A}\cap\underset{\otimes}{B}=\phi$, $\underset{\otimes}{A}$ and $\underset{\otimes}{B}$ are mutually exclusive; when $\underset{\otimes}{A}\cap\underset{\otimes}{B}=\phi$, sum event $\underset{\otimes}{A}\cup\underset{\otimes}{B}$ is written in $\underset{\otimes}{A}\oplus\underset{\otimes}{B}$, and if any two events of a group of events $\underset{\otimes}{A}1$, $\underset{\otimes}{A}2$ ...are mutually exclusive, each two events are said to be exclusive in the group, and to the group events $\underset{\otimes}{A}1$, $\underset{\otimes}{A}2$, ... exclusive in pairs, sum events $\underset{\otimes}{A}_1\cup\underset{\otimes}{A}_2$ is recorded as $\underset{\otimes}{A}_1\oplus\underset{\otimes}{A}_2$,...

DEFINITION 6 If the two events $\underset{\otimes}{A}$, $\underset{\otimes}{B}$ satisfy the conditions $\underset{\otimes}{A}\cup\underset{\otimes}{B}=U$, $\underset{\otimes}{A}\cap\underset{\otimes}{B}=\phi$, events A and B are said to be inverse or opposite, eventA is called the inverse event of event B (or opposite event, or co-event).

THEOREM: If $\underset{\otimes}{A}$ and $\underset{\otimes}{B}$ are mutually inverse, i.e. $\underset{\otimes}{A}\cup\underset{\otimes}{B}=U$, $\underset{\otimes}{A}\cap\underset{\otimes}{B}=\phi$, $\underset{\otimes}{A}$ and $\underset{\otimes}{B}$ are Cantor's subsets, i.e. the subordinate function value comes from set {o,1}

Proof: Since $\underset{\otimes}{A}\cup\underset{\otimes}{B}=U$ and $\underset{\otimes}{A}\cap\underset{\otimes}{B}=\phi$, therefore

$$\overline{u}_{\underset{\otimes}{A}}(x)\vee_{\underset{\otimes}{B}}\overline{u}\ (x)=1,\ \underline{u}_{\underset{\otimes}{A}}(x)\vee_{\underset{\otimes}{B}}\underline{u}\ (x)=1$$

$$\overline{u}_{\underset{\otimes}{A}}(x)\wedge\overline{u}_{\underset{\otimes}{B}}(x)=0,\ \underline{u}_{\underset{\otimes}{A}}(x)\wedge\underline{u}_{\underset{\otimes}{B}}(x)=0,\ x\in U$$

Provided that $\underline{u}_{\underset{\otimes}{A}}(x)=1$, $\overline{u}_{\underset{\otimes}{A}}(x)=\underline{u}_{\underset{\otimes}{A}}(x)=1$, $\overline{u}_{\underset{\otimes}{B}}(x)=\underline{u}_{\underset{\otimes}{B}}(x)=0$, their subordinate functon value belongs to set {0,1}, which is Cantor's set

II. The Axiom System of Grey Probability

The introduction of grey probability is under two conditions:(i) the elements of basic space U are finite or infinitebut countable, and (ii) the elements of U are contiuous type and infinite. However, we here mainly discuss the first condition.

It is pointed out in Literature [1] that grey subsets of U fall into three classes: (i) Cantor's subset, (ii) fuzzy subset, and (iii) real grey subset.

First of all we introduce to Cantor' subset the probability of classical mathematics.

DEFINITION 1: Let the definition domain of P(A) be a set, which is composed of Cantor's subset of U, satisfying the following conditions:

(i) $0\le P(A)\le 1$ for any event A,

(ii) $P(U)=1$, $P(\phi)=0$,

(iii) $P(A_1+A_2+\ldots)=P(A_1)+P(A_2)+\ldots$ is true for the inverse-each-other random events $A_1, A_2, \ldots$, in which $A_i\cap A_j=\phi\ (i\neq j)$, P(A) is called probability of event A.

This definition is completely classical probability built on Cantor's set in classical mathematics.

DEFINITION 2: Let $U=\{e_1, e_2, \ldots e_n, \ldots\}$

$$\underset{\otimes}{A}=\frac{(\overline{u}_{\underset{\otimes}{A}}(e_{i_1}),\underline{u}_{\underset{\otimes}{A}}(e_{i_1}))}{e_{i_1}}+\frac{(\overline{u}_{\underset{\otimes}{A}}(e_{i_2}),\underline{u}_{\underset{\otimes}{A}}(e_{i_2}))}{e_{i_2}}+\ldots$$

and probability of grey set $\underset{\otimes}{A}$ is defined as

$$P(A)=[\underline{u}_{\underset{\otimes}{A}}(e_{i_1})P(\{e_{i_1}\})+\underline{u}_{\underset{\otimes}{A}}(e_{i_2})P(\{e_{i_2}\})+\ldots+\overline{u}_{\underset{\otimes}{A}}(e_{i_1})P(\{e_{i_1}\})+\overline{u}_{\underset{\otimes}{A}}(e_{i_2})P(\{e_{i_2}\})$$

where "..." means two (i) countable infinite terms, (ii) the finite terms.

THEOREM 1: When $\underset{\otimes}{A}$ is a Cantor's subset, and $\underset{\otimes}{A}=\{e_{i_1}, e_{i_2}, \ldots\}$, and since $\bar{u}(e_k)=\underline{u}(e_k)=1$, therefore $P(\underset{\otimes}{A})=P(e_{i_1})+P(e_{i_2})+\dots$, which is identical with the probability in Definition 1

EXAMPLE: Let $u=\{a,b\}$, $\underset{\otimes}{A}=\frac{(1,\frac{1}{2})}{a}+\frac{(\frac{1}{2},\frac{1}{3})}{b}$ $P(\{a\})=\frac{1}{4}$ $P(\{b\})=\frac{3}{4}$

then $P(\underset{\otimes}{A})=[\underline{u}_{\underset{\otimes}{A}}(a)P(\{a\})+\underline{u}_{\underset{\otimes}{A}}(b)P(\{b\}),\ \bar{u}_{\underset{\otimes}{A}}(a)P(\{a\})+\bar{u}_{\underset{\otimes}{A}}(b)P(\{b\})]$

$=[\frac{1}{2}\times\frac{1}{4}+\frac{1}{3}\times\frac{3}{4}, 1\times\frac{1}{4}+\frac{1}{2}\times\frac{3}{4}]=[\frac{3}{8},\frac{5}{8}]$

III. Properties of Grey Probability

Property 1: $P(U)=1$, $P(\phi)=0$

Proof: From Theorem 1 in Part II, we obtain $P(U)=1$, $P(\phi)=0$

Property 2: If $P(A)=[a,b,]$, then

$$0\le\min\{a,b\}\le\max\{a,b\}\le 1,\ \text{i.e. } 0\le a\le b\le 1.$$

Proof: Let $\underset{\otimes}{A}=\frac{(\bar{u}_{\underset{\otimes}{A}}(e_{i_1}),\ \underline{u}_{\underset{\otimes}{A}}(e_{i_1}))}{e_{i_1}}+\frac{(\bar{u}_{\underset{\otimes}{A}}(e_{i_2}),\ \underline{u}_{\underset{\otimes}{A}}(e_{i_2}))}{e_{i_2}}+\dots$, then

$b=\bar{u}_{\underset{\otimes}{A}}(e_{i_1})P(\{e_{i_1}\})+\bar{u}_{\underset{\otimes}{A}}(e_{i_2})P(\{e_{i_2}\})+\dots$

$a=\underline{u}_{\underset{\otimes}{A}}(e_{i_1})P(\{e_{i_1}\})+\underline{u}_{\underset{\otimes}{A}}(e_{i_2})P(\{e_{i_2}\})+\dots$

Since each term of a or b additioned is non-negative, therefore $0\le\min\{a,b\}$; since $\max\{a,b\}=b\le P(\{e_{i_1}\})+P(\{e_{i_2}\})+\dots\le 1$, then $0\le\min\{ab\}\le\max\{a,b\}\le 1$

Property 3: If $\underset{\otimes}{A}\cap\underset{\otimes}{B}=\phi$, then $P(A+b)=P(A)+P(B)$

Proof: If $\underset{\otimes}{A}=\frac{(\bar{u}_{\underset{\otimes}{A}}(e_1),\ \underline{u}_{\underset{\otimes}{A}}(e_1))}{e_1}+\frac{(\bar{u}_{\underset{\otimes}{A}}(e_2),\ \underline{u}_{\underset{\otimes}{A}}(e_2))}{e_2}+\dots$ and

$\underset{\otimes}{B}=\frac{(\bar{u}_{\underset{\otimes}{B}}(e_1),\ \underline{u}_{\underset{\otimes}{B}}(e_1))}{e_1}+\frac{(\bar{u}_{\underset{\otimes}{B}}(e_2),\ \underline{u}_{\underset{\otimes}{B}}(e_2))}{e_2}+\dots$ are given, and as

$\underset{\otimes}{A}\wedge\underset{\otimes}{B}=\phi$ and $\bar{u}_{\underset{\otimes}{A}\wedge\underset{\otimes}{B}}=\bar{u}_{\underset{\otimes}{A}}\wedge\bar{u}_{\underset{\otimes}{B}}$, $\underline{u}_{\underset{\otimes}{A}\wedge\underset{\otimes}{B}}=\underline{u}_{\underset{\otimes}{A}}\wedge\underline{u}_{\underset{\otimes}{B}}$, thus $\bar{u}_{\underset{\otimes}{A}\wedge\underset{\otimes}{B}}\equiv 0$, $\underline{u}_{\underset{\otimes}{A}\wedge\underset{\otimes}{B}}\equiv 0$ which provide that $\bar{u}_{\underset{\otimes}{B}}(e_{i_k})=\underline{u}_{\underset{\otimes}{B}}(e_{i_k})=0$, $(k=1,2,\dots)$ if $\bar{u}_{\underset{\otimes}{A}}(e_{i_k})\ne 0$ $(k=1,2,\dots)$.

It is accepted that $\bar{u}(x)=0$ can be omitted in the expression of fraction, which has two cases as follows

(i) The number of i_K satisfying $u(e_{i_k})\ne 0$ is finite, which can be assumed as $1,2,\dots n$, therefore

$$\underset{\otimes}{A}=\frac{(\bar{u}_{\underset{\otimes}{A}}(e_1),\ \underline{u}_{\underset{\otimes}{A}}(e_1))}{e_1}+\dots+\frac{(\bar{u}_{\underset{\otimes}{A}}(e_{n_0}),\ \underline{u}_{\underset{\otimes}{A}}(e_{n_0}))}{e_{n_0}}$$

$$\underset{\otimes}{B}=\frac{(\bar{u}_{\underset{\otimes}{B}}(e_{n_0+1}),\ \underline{u}_{\underset{\otimes}{B}}(e_{n_0+1}))}{e_{n_0+1}}$$

$$\underset{\otimes}{A}\cup\underset{\otimes}{B}=\underset{\otimes}{A}+\underset{\otimes}{B}=\frac{(\bar{u}_{\underset{\otimes}{A}}(e_1),\ \underline{u}_{\underset{\otimes}{A}}(e_1))}{e_1}\dots+\frac{(\bar{u}_{\underset{\otimes}{A}}(e_{n_0}),\ \underline{u}_{\underset{\otimes}{A}}(e_{n_0}))}{e_{n_0}}+\frac{(\bar{e}_{\underset{\otimes}{B}}(e_{n_0+1}),\ \underline{e}_{\underset{\otimes}{B}}(e_{n_0+1}))}{e_{n_0+1}}$$

and then

$$P(\underset{\otimes}{A}+\underset{\otimes}{B}) = [\ \underline{u}_{\underset{\otimes}{A}}(e_1)P(\{e_1\}) + \ldots + \underline{u}_{\underset{\otimes}{A}}(e_{n_0})P(\{e_{n_0}\}) + \underline{u}_{\underset{\otimes}{B}}(e_{n_0+1})P(\{e_{n_0+1}\})$$

$$+ \ldots,\ \overline{u}_{\underset{\otimes}{A}}(e_1)P(\{e_1\}) + \ldots + \overline{u}_{\underset{\otimes}{A}}(e_{n_0})P(\{e_{n_0}\})$$

$$+ \overline{u}_{\underset{\otimes}{B}}(e_{n_0+1})P(\{e_{n_0+1}\}) + \ldots\ ]$$

$$= [\ \underline{u}_{\underset{\otimes}{A}}(e_1)P(\{e_1\}) + \ldots + \underline{u}_{\underset{\otimes}{A}}(e_{n_0})P(\{e_{n_0}\}),$$

$$\overline{u}_{\underset{\otimes}{A}}(e_1)P(\{e_1\}) + \ldots + \overline{u}_{\underset{\otimes}{A}}(e_{n_0})P(\{e_{n_0}\})]$$

$$+ [\underline{u}_{\underset{\otimes}{B}}(e_{n_0+1})P(\{e_{n_0+1}\}) + \ldots,\ \overline{u}_{\underset{\otimes}{B}}(e_{n_0+1})P(\{e_{n_0+1}\})] = P(\underset{\otimes}{A}) + P(\underset{\otimes}{B})$$

$P(\underset{\otimes}{A}+\underset{\otimes}{B})=P(\underset{\otimes}{A})+P(\underset{\otimes}{B})$ can be proved when the number of i_k in $\overline{u}_{\underset{\otimes}{A}}(e_{i_k})\neq 0$ is finite.

(ii) When the number of e_{i_k} in $\overline{u}_{\underset{\otimes}{A}}(e_{i_k})\neq 0$ and $\overline{u}_{\underset{\otimes}{B}}(e_{i_k})\neq 0$ is infinite, $u_{\underset{\otimes}{A}}(e_{i_k})=U_{\underset{\otimes}{B}}(e_{i_k})=0$ can be omitted, and , to the conclusion generalized, let

$$A= \frac{(\overline{u}_{\underset{\otimes}{A}}(e_1),\ \underline{u}_{\underset{\otimes}{A}}(e_1))}{e_1} + \frac{(\overline{u}_{\underset{\otimes}{A}}(e_3),\ \underline{u}_{\underset{\otimes}{A}}(e_3))}{e_3} + \ldots + \frac{(\overline{u}_{\underset{\otimes}{A}}(e_{2n+1}),\ \underline{u}_{\underset{\otimes}{A}}(e_{2n+1}))}{e_{2n+1}} + \ldots$$

$$B= \frac{(\overline{u}_{\underset{\otimes}{B}}(e_2),\ \underline{u}_{\underset{\otimes}{B}}(e_2))}{e_2} + \frac{(\overline{u}_{\underset{\otimes}{B}}(e_4),\ \underline{u}_{\underset{\otimes}{B}}(e_4))}{e_4} + \ldots + \frac{(\overline{u}_{\underset{\otimes}{B}}(e_{2n}),\ \underline{u}_{\underset{\otimes}{B}}(e_{2n}))}{e_{2n}} + \ldots$$

then

$$P(\underset{\otimes}{A}+\underset{\otimes}{B}) = [\ \underline{u}_{\underset{\otimes}{A}}(e_1)P(\{e_1\}) + \underline{u}_{\underset{\otimes}{B}}(e_2)P(\{e_2\}) + \ldots + \overline{u}_{\underset{\otimes}{A}}(e_1)P(\{e_1\})$$

$$+ \overline{u}_{\underset{\otimes}{B}}(e_2)P(\{e_2\}) + \ldots]$$

$$= [\ \underline{u}_{\underset{\otimes}{A}}(e_1)P(\{e_1\}) + \ldots + \underline{u}_{\underset{\otimes}{A}}(e_{2n+1})P(\{e_{2n+1}\}) + \ldots,$$

$$\overline{u}_{\underset{\otimes}{A}}(e_1)P(\{e_1\}) + \ldots + \overline{u}_{\underset{\otimes}{A}}(e_{2n+1})P(\{e_{2n+1}\}) + \ldots\ ]$$

$$+ [\ \underline{u}_{\underset{\otimes}{B}}(e_2)P(\{e_2\}) + \ldots + \underline{u}_{\underset{\otimes}{B}}(e_{2n})P(\{e_{2n}\}) + \ldots\ ]$$

$$= P(\underset{\otimes}{A}) + P(\underset{\otimes}{B}),$$

thus Property 3 is proved by combining (i) with (ii).

THEOREM 1: If any two grey events of $\underset{\otimes}{A}_1, \underset{\otimes}{A}_2, \ldots \underset{\otimes}{A}_n$ are mutually exclusive, then $P(\underset{\otimes}{A}_1)+P(\underset{\otimes}{A}_2)+\ldots+P(\underset{\otimes}{A}_n) = P(\underset{\otimes}{A}_1+\underset{\otimes}{A}_2+\ldots+\underset{\otimes}{A}_n)$.

Proof: By using mathematical induction,

(i) when n=2, Theorem 1 is proved by Property 3;

(ii) when n=k, $P(\underset{\otimes}{A}_1+\ldots+\underset{\otimes}{A}_k)=P(\underset{\otimes}{A}_1)+\ldots+P(\underset{\otimes}{A}_k)$ occurs $\underset{\otimes}{A}_i\cap\underset{\otimes}{A}_j=\phi$ $(i\neq j)$, $i,j=1,2,\ldots n$. Therefore

$$(\underset{\otimes}{A}_1+\underset{\otimes}{A}_2+\ldots+\underset{\otimes}{A}_k)\cap\underset{\otimes}{A}_{k+1} = (\underset{\otimes}{A}_1\cap\underset{\otimes}{A}_{k+1})+(\underset{\otimes}{A}_2\cap\underset{\otimes}{A}_{k+1})+\ldots+(\underset{\otimes}{A}_k\cap\underset{\otimes}{A}_{k+1})$$

$=\phi+\phi+\ldots+\phi=\phi$, and

$$P(\underset{\otimes}{A}_1 + \dots + \underset{\otimes}{A}_K + \underset{\otimes}{A}_{K+1}) = P(\underset{\otimes}{A}_1)+P(\underset{\otimes}{A}_2)+\dots+P(\underset{\otimes}{A}_K)+P(\underset{\otimes}{A}_{K+1})$$

From (i) and (ii) we obtain, for the arbitrary naturual number n $P(\underset{\otimes}{A}_1 + \dots + \underset{\otimes}{A}_n) = P(\underset{\otimes}{A}_1) + \dots + P(\underset{\otimes}{A}_n)$.

DEFINITION: Define $\lim\limits_{n\to\infty} [a_n, b_n] = [\lim\limits_{n\to\infty} a_n, \lim\limits_{n\to\infty} b_n]$ for the sequence $[a_n, b_n]$, n=1,2,... of grey number.

DEFINITION: $\sum\limits_{n=1}^{\infty} [a\ , b\ ] = [a_1, b_1] + [a_2, b_2] + \dots + [a_n, b_n] + \dots$ is called grey series, and $\underset{\otimes}{S}_n = \sum\limits_{i=1}^{n} [a_i, b_i]$ is the sum of the first n terms of grey series with $\underset{\otimes}{S} = \lim\limits_{n\to\infty} \underset{\otimes}{S}_n$ defined as the sum of the series. If $\lim\limits_{n\to\infty} \underset{\otimes}{S}_n$ exists, the series converges, otherwise the series diverges.

THEOREM 2: If basic space U is a finite set, $\underset{\otimes}{A}_i \cap \underset{\otimes}{A}_j = \phi$ $(i \neq j)$ j=1,2,3,..., $\underset{\otimes}{A}_1, \underset{\otimes}{A}_2$, ...are grey events, then $P(\underset{\otimes}{A}_1 + \underset{\otimes}{A}_2 + \dots) = P(\underset{\otimes}{A}_1) + P(\underset{\otimes}{A}_2) + \dots$.

Proof: Since U is a finite set, we can assume $U=\{e_1, e_2, \dots, e_n\}$, $n \in R$, and since $\underset{\otimes}{A}_1, \underset{\otimes}{A}_2, \dots, \underset{\otimes}{A}_n$ ..., (i) is a group of grey countable events, and $\underset{\otimes}{A}_i \cap \underset{\otimes}{A}_j = \phi (i \neq j)$. And for each e_i (i=1,2,...,n) there is at most one $\underset{\otimes}{A}_{n_i}$, so $u_{\underset{\otimes}{A}_{n_i}}(e_i) \neq 0$. Otherwise we can deduce that any two events of the set of grey events are not mutually exclusive, and contradictary to hypothesi. Therefore we conclude that there exist no more than n grey subsets $\underset{\otimes}{A}_{n_1}$, $\underset{\otimes}{A}_{n_2}$,..., $\underset{\otimes}{A}_{n_K}$ $(k \in \{1,2,\dots,n\})$ in (i), which are not empty sets but the rest are all ϕ. Thus

$$P(\underset{\otimes}{A}_1 + \underset{\otimes}{A}_2 + \dots) = P(\underset{\otimes}{A}_{n_1} + \underset{\otimes}{A}_{n_2} + \dots \underset{\otimes}{A}_{n_K}) = P(\underset{\otimes}{A}_{n_1}) + P(\underset{\otimes}{A}_{n_2}) + \dots + P(\underset{\otimes}{A}_{n_K}) =$$
$$P(\underset{\otimes}{A}_{n_1}) + \dots + P(\underset{\otimes}{A}_{n_K}) + P(\phi) + P(\phi) + \dots = P(\underset{\otimes}{A}_1) + P(\underset{\otimes}{A}_2) + \dots$$

IV. Relations among Grey Probability, fuzzy Probability and Classical Probability

Through the knowledge of "theory of grey set", "grey number" and the definitions of probability of classical mathematics and fuzzy mathematics, we can deduce that when grey events are limited to Cantor's subset of basic space U, grey probability and classical probability are identical with each other; when grey events are limited to fuzzy subset of U, grey probability and fuzzy probability are identical. Therefore grey probability is the expansion of classical probability and fuzzy probability, and also they are the special cases of grey probability. The conclusion deduced from axiom system of grey probability can be used in classical probabilty and fuzzy probability without any condition. Otherwise, careful considerations should be made.

References:

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[3] LiuKai Di, Wang Qing Yin, Wu He Qin, The Operations of Grey Number and its Operational Properties