ON THE DEGREE OF FUZZINESS OF A FUZZY SET

Eugene Roventa
Department of Computer Science, Glendon College, York University
2275 Bayview Ave. Toronto, Ont. M4N 3M6
Canada

ABSTRACT: The purpose of this paper is to study the notion of transom fuzzy indefinite integral on the σ-algebra $\tilde{\mathcal{M}}(X,\beta)$, of fuzzy measurable subsets of a given space X. The probability of a fuzzy event $\tilde{A}$, defined by a Lebesque-Stieltjes integral introduced in [1] is obtained as a particular case (Remark 3). These measures are used to distinguish fuzzy sets which have the same fuzzy degree (Definition 4). Thus we introduce two parameters for characterizing the fuzziness of a fuzzy set. For fundamental notions concerning measure and integration theory see [2], and for terminology connected to measures on fuzzy sets see [3] and [4].

KEYWORDS: fuzzy sets, transom, support, measure of fuzziness, measure theory.

INTRODUCTION: If X is an arbitrary set and $\tilde{F} \subseteq \tilde{1}$ is a fixed fuzzy set then we shall call transom (or esssential part) of the fuzzy set $\tilde{F}$, the set (in the usual sense) $T(\tilde{F}) = \{x \in X | 0<\tilde{F}(x)<1\}$ where $\tilde{F} \in [0,1]^X$. The properties of the function $\tilde{F} \mapsto T(\tilde{F})$ are studied in [4]. In [3] there are also introduced the usual sets: $S(\tilde{F})=\{x \in X | \tilde{F}(x)>0\}$, $Z(\tilde{F})=\{x \in X | \tilde{F}(x) =0\}$ and $H(\tilde{F})=\{x \in X | \tilde{F}(x)=1\}$.

Let (X,β,μ) be a measure space where β is a σ-algebra and $\mu:\beta\to\mathbb{R}_+$ is a positive finite measure. We consider further the σ-algebra $\tilde{\mathcal{M}}(X,\beta)$ of the fuzzy measurable sets from X and a β-measurable function $f:X\to\mathbb{R}$. In [3] we studied the transom measure associated to μ as the function $m_T:\tilde{\mathcal{M}}(X,\beta)\to\mathbb{R}_+$ defined by:

$$m_T(\tilde{F}) = \mu(T(\tilde{F})).$$

DEFINITION 1. We call transom fuzzy indefinite integral of f, corresponding to the usual measure μ, the function $I_f:\tilde{\mathcal{M}}(X,\beta)\to\mathbb{R}$ defined by:

$$I_f(\tilde{F})=\int f\tilde{F}T(\tilde{F})d\mu.$$

DEFINITION 2.[6] We call support fuzzy indefinite integral of f, corresponding to the usual measure μ the function $\bar{I}_f:\tilde{\mathcal{M}}(X,\beta)\to\mathbb{R}$ defined by:

$$\bar{I}_f(\tilde{F})=\int f\tilde{F}d\mu.$$

REMARK 1. If $\tilde{F}$ is an usual set, then $T(\tilde{F})=\phi$ hence $I_f(\tilde{F})=0$ and $\bar{I}_f(\tilde{F})$ is the usual indefinite integral. If $f=1$, we shall denote:

$$I_1(\tilde{F})=\int\tilde{F}T(\tilde{F})d\mu=\int_{T(\tilde{F})}\tilde{F}d\mu$$

$$\bar{I}_1(\tilde{F})=\int\tilde{F}d\mu=\int_{S(\tilde{F})}\tilde{F}d\mu\ .$$

DEFINITION 3. If $\tilde{F}_1,\tilde{F}_2\in\tilde{\mathcal{M}}(X,\beta)$ we shall designate $\tilde{F}_1\sim\tilde{F}_2 \Leftrightarrow m_T(\tilde{F}_1)=m_T(\tilde{F}_2)$ where m_T is the transom measure generated by μ [3]. Obviously "~" is an equivalence relation. We shall denote the corresponding quotient set with $\mathcal{C}_T$.

DEFINITION 4. We call fuzzy degree any equivalence class from $\mathcal{C}_T$.

REMARK 2. The transom measure $m_T: \tilde{\mathcal{M}}(X,\beta) \to \mathbb{R}$ studied in [3] characterizes the fuzzy degree of a fuzzy set and it is the same for all the fuzzy sets whose transom have the same measure. Using the function I_1 we can distinguish the fuzzy sets which have the same fuzzy degree.

REMARK 3. If $\mu(X)=1$ and if the integral is the Lebesque-Stieltjes integral, then $\bar{I}_1$ coincides whith the probability of a fuzzy event introduced in [1].

PROPOSITION 1. The following inequalities hold:

$$I_1(\tilde{F}) < m_T(\tilde{F})$$
$$\bar{I}_1(\tilde{F}) < m_S(\tilde{F})$$

where $m_S = \mu(S(\tilde{F}))$ [3].

The assertion follows from the fact that $\tilde{F}(x) < 1$, for every $x \in X$.

PROPOSITION 2. The following relation holds:

$$\bar{I}_1(\tilde{F}) = I_1(\tilde{F}) + \mu[H(\tilde{F})].$$

It results from: $S(\tilde{F}) = T(\tilde{F}) \cup H(\tilde{F})$.

REMARK 4. Denoting $V(\tilde{F}) = \dfrac{I_1(\tilde{F})}{\bar{I}_1(\tilde{F})}$ and $W(\tilde{F}) = \dfrac{\mu[H(\tilde{F})]}{\bar{I}_1(\tilde{F})}$

we have $V(\tilde{F}) + W(\tilde{F}) = 1$ whatever would be $\tilde{F} \in \tilde{\mathcal{M}}(X,\beta)$ with $\bar{I}_1(\tilde{F}) \neq 0$.
If V is small for a certain fuzzy degree, $\tilde{F}$ is "almost usual set" and when W is small for a certain fuzzy degree, then $\tilde{F}$ is an "almost proper fuzzy set".

DEFINITION 5[2] We say that the β-measurable function $f:X \to \mathbb{R}$ $(f=f^{+}-f^{-})$ has an integral related the positive measure μ if at least one of the function f^{+},f^{-} is integrable that is either $\int f^{+}d\mu < \infty$ or $\int f^{-}d\mu < \infty$.

THEOREM 1. If f has an integral with respect to the measure μ, then the transom fuzzy indefinite integral of f (resp. the support fuzzy indefinite integral of f) is a signed measure on $\tilde{\mathcal{M}}(X,\beta)$.

Proof. Because $f = f^{+} - f^{-}$ it is sufficient to consider the case $f \geq 0$. Let $\tilde{F}=\bigcup_{k=1}^{\infty}\tilde{F}_k$, $\tilde{F}_k \cap \tilde{F}_{k'} = \tilde{0}$, $k\neq k'$, $\tilde{F}_k \in \tilde{\mathcal{M}}(X,B)$. It is clear$\left(T(\tilde{F}_k) \cap T(\tilde{F}_{k'})=\phi\right)$ that $\tilde{F}(x)= \sum_{k=1}^{\infty} \tilde{F}_k(x)$ and $T(\tilde{F})(x) = \sum_{k=1}^{\infty} T(\tilde{F}_k)(x)$ where $T(\tilde{F})(x) =1$ if $x\in T(\tilde{F})$ and $T(\tilde{F})(x)=0$ otherwise. One can observe that whichever would be $x\in X$ there is an unique k_0 such that $\tilde{F}_{k_0}(x)\neq 0$ (resp. $T(\tilde{F}_{k_0})(x)\neq 0$). Hence $\tilde{F}(x)T(\tilde{F})(x)= =\sum_{k=1}^{\infty} \tilde{F}(x)T(\tilde{F}_k)(x)$ holds for $x\in X$. Multiplying the above relation with $f\geq 0$ and integrating we get $I_f(\bigcup_{k=1}^{\infty} \tilde{F}_k) = \sum_{k=1}^{\infty} I_f(\tilde{F}_k)$. For the support fuzzy indefinite integral the proof is analogous and use the support instead of the transom.

REMARK 5. If $f\geq 0$ then both I_f and $\bar{I}_f$ are nonegative measures. Of course this remark is valid for I_1 resp. $\bar{I}_1$. But I_1 (unlike $\bar{I}_1$) is neither monotonic nor subtractive (this is a consequence of the fact that for an arbitrary fuzzy set $\tilde{A}$ does not hold $\tilde{A} \cap \complement\tilde{A} = \phi$). The next theorem is a generalisation of the countable additivity of the transom fuzzy indefinite integral.

THEOREM 2. If f has an integral with respect to the measure μ then for each sequence $\{\tilde{F}_k\}_{k\in\mathbb{N}}$ the following equalities hold:

(i) $I_f(\bigcup_{k=1}^{\infty} \tilde{F}_k)=\sum_{k=1}^{\infty} I_f(\tilde{F}_k)$ whenever $H(\bigcup_{k=1}^{\infty} \tilde{F}_k)\subseteq \bigcap_{k=1}^{\infty} H(\tilde{F}_k)$ and $m_T(\tilde{F}_k \cap \tilde{F}_{k'})=0, k\neq k'$.

(ii) $\bar{I}_f(\bigcup_{k=1}^{\infty} \tilde{F}_k)=\sum_{k=1}^{\infty} \bar{I}_f(\tilde{F}_k)$ if $m_S(\hat{F}_k \cap \hat{F}_{k'})=0,\ k\neq k'$.

Proof (i) If we designate $\tilde{F}= \bigcup_{k=1}^{\infty} \tilde{F}_k$, then $H(\bigcup_{k=1}^{\infty} \tilde{F}_k)\subseteq \bigcap_{k=1}^{\infty} H(\tilde{F}_k)$ implies: $T(\bigcup_{k=1}^{\infty} \tilde{F}_k)= =\bigcup_{k=1}^{\infty} T(\hat{F}_k)$. Hence we can write: $T(\tilde{F}) = A_1 \cup A_2 \cup \ldots \cup A_k \cup \ldots,\ A_k \cap A_{k'}=\phi$, (for $k\neq k'$) where $A_1=T(\tilde{F}_1)$, $A_k=T(\tilde{F}_k)\setminus\bigcup_{m=1}^{k-1} [T(\tilde{F}_k)\cap T(\hat{F}_n)]$. We take by definition $\tilde{E}_k=\tilde{F}_k \cap A_k\in \tilde{\mathcal{M}}(X,\beta)$ and one can observe that $T(\tilde{E}_k)=A_k$. Therefore: $T(\tilde{F})= \bigcup_{k=1}^{\infty} T(\tilde{E}_k)$ with $T(\tilde{E}_k) \cap T(\tilde{E}_{k'})=\phi$ $(k\neq k')$. By hypothesis we have $H(\tilde{F}_i)=H(\hat{F}_j)$, $i\neq j$ hence $0=m_T(\hat{F}_k \cap \tilde{F}_{k'}) = \mu[T(\tilde{F}_k \cap \tilde{F}_{k'})]= \mu[T(\tilde{F}_k)\cap \cap T(\tilde{F}_{k'})]$. But $T(\tilde{E}_k)=T(\tilde{F}_k)\setminus\bigcup_{m=1}^{k-1} [T(\hat{F}_k) \cap T(\tilde{F}_n)]$ therefore $m_T(\tilde{E}_k)=m_T(\tilde{F}_k)$ wherefrom $\int_{T(\tilde{E}_k)} f\tilde{E}_k d\mu = \int_{T(\tilde{F}_k)} f\tilde{F}_k d\mu$ and $I_f(\bigcup_{k=1}^{\infty} \tilde{F}_k)= \sum_{k=1}^{\infty} \int_{T(\tilde{E}_k)} f\tilde{E}_k d\mu= \sum_{k=1}^{\infty} \int_{T(\tilde{F}_k)} f\tilde{F}_k d\mu= \sum_{k=1}^{\infty} I_f(\tilde{F}_k)$.

For $\bar{I}_f$ one can judge in an analogous manner using $S(\tilde{F})$.

THEOREM 3. If the function f has an integral related to μ then for every increasing sequence $\{\tilde{F}_k\}_{k\in\mathbb{N}}$ of fuzzy sets from $\tilde{\mathcal{M}}(X,\beta)$ with $\tilde{F}_k\nearrow\tilde{F}$, the following inequalities hold :

(i) $I_f(\tilde{F})= \lim_{k\to\infty} I_f(\hat{F}_k)$ if $H(\tilde{F}_k)= H(\hat{F})$, $k\in\mathbb{N}$

(ii) $\bar{I}_f(\tilde{F})= \lim_{k\to\infty} \bar{I}_f(\tilde{F}_k)$.

Proof (i) If $\tilde{F}_k \nearrow \tilde{F}$ and if $H(\tilde{F}_k)=H(\tilde{F})$, $k\in N$, we have $T(\tilde{F}_k) \nearrow T(\tilde{F})$, hence
$T(\tilde{F}) = \bigcup_{k=1}^{\infty} T(\tilde{F}_k) = T(\tilde{F}_1) \cup [T(\tilde{F}_2) \setminus T(\tilde{F}_1)] \cup \ldots \cup [T(\tilde{F}_n) \setminus T(\tilde{F}_{n-1})] \cup \ldots$
If we put by definition $\tilde{F}_o = \tilde{0}$, we get:

$$I_f(\tilde{F}) = \int_{T(\tilde{F})} f\tilde{F}d\mu = \sum_{n=1}^{\infty} \int_{T(\tilde{F}_n)-T(\tilde{F}_{n-1})} f\tilde{F}d\mu = \lim_{k\to\infty} \sum_{n=1}^{k} \int_{T(\tilde{F}_n)-T(\tilde{F}_{n-1})} f\tilde{F}d\mu = \lim_{k\to\infty} \int_{T(\tilde{F}_k)} f\tilde{F}d\mu =$$

$$= \lim_{k\to\infty} I_f(\tilde{F}_k)$$

(ii) can be proved in an analogous manner by replacing in the above proof the transom by the support.

THEOREM 4. $\bar{I}_1$ is a complete measure on $\tilde{\mathcal{M}}(X,\beta)$ iff μ is a complete measure on β.

Proof We suppose that $\bar{I}_1(\tilde{F})=0$. Because $\tilde{F}(x)>0$ if $x\in S(\tilde{F})$ we get $\mu[S(\tilde{F})]=0$. Let us consider $\tilde{E}\subseteq\tilde{F}$; then $S(\tilde{E})\subseteq S(\tilde{F})$. But μ is assumed to be complete, hence $S(\tilde{E})\in\beta$ and $\mu[S(\tilde{E})] = 0$. If we consider $E_\alpha=\{x|\tilde{F}(x)\leq\alpha\}$ (for $\alpha\in[0,1]$) then $E_\alpha\subseteq S(\tilde{E})$ and we get (from the completness of μ) that $E_\alpha\in\beta$ ($\alpha\in(0,1]$). Further the set $E_o = \{x|\tilde{E}(x)= 0\}=X\setminus S(\tilde{E})$, hence $E_o\in\beta$. Therefore $E_\alpha\in\beta$ whichever would $\alpha\in[0,1]$ be. But this means, by definition that $\tilde{E}\in\tilde{\mathcal{M}}(X,\beta)$. The converse implication follows since $\bar{I}_1$ is an extension of μ.

REMARK 6. The completness of the measure I_1 would imply $\tilde{\mathcal{M}}(X,\beta)=[0,1]^X$ (this is so because we have $I_1(\tilde{1})=0$).

DEFINITION 6. Let m,n be two signed finite measures given on $\tilde{\mathcal{M}}(X,\beta)$. We shall say that m is absolutely continous with respect to the measure n, if for every $\varepsilon>o$, there is a $\delta>o$, such that: $\tilde{E}\subseteq\tilde{1}$, $\bar{n}(\tilde{E})<\delta$ imply $|m(\tilde{E})|<\varepsilon$.

REMARK 7. $\bar{n}$ is the total variation of n, which in the case $n\geq 0$ coincides with n (on $\tilde{\mathcal{M}}(X,\beta)$).

REMARK 8. If μ is positive and finite on β, then, I_1 and $\bar{I}_1$ are positive and finite measures on $\tilde{\mathcal{M}}(X,\beta)$.

THEOREM 5. If $f:X\to\mathbb{R}$ is integrable, then I_f (resp.$\bar{I}_f$) *is* absolutely continous with respect to I_1 (resp.$\bar{I}_1$).

Proof We will take first $f\geq 0$. There is a sequence $\{f_n\}_{n\in N}$ of simple functions, with $f_n\leq n$, such that:

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu < \infty.$$

For $\varepsilon>0$, there is $n_o\in N$ such that for $n\geq n_o$:

$$\int f d\mu - \int f_n d\mu < \varepsilon/2 .$$

Let us consider the difference:

$$I_f(\tilde{E}) - I_{f_n}(\tilde{E}) = I_{f-f_n}(\tilde{E}) = \int (f-f_n)\tilde{E}T(\tilde{E})d\mu \leq \int (f-f_n)d\mu \leq \varepsilon/2 \text{ (for } n\geq n_o).$$

Hence:

$$I_f(\tilde{E}) < I_{f_{n_o}}(\tilde{E}) + \varepsilon/2 .$$

But:

$$I_{f_{n_o}}(\tilde{E}) = \int f_{n_o}\tilde{E}T(\tilde{E})d\mu \leq n_o \int_{T(\tilde{E})} \tilde{E} d\mu = n_o I_1(\tilde{E}) .$$

Let $\tilde{E}\in\tilde{\mathcal{M}}(X,\beta)$ such that $I_1(\tilde{E})<\varepsilon/2n_o$; then we get: $I_f(\tilde{E}) < \varepsilon$.

If f is arbitrary (i.e. does not keep the same sign) integrable function, we consider f^+, f^- and because $|f|=f^+ + f^-$ we deduce that $|f|$ is also integrable. Further we apply the well known inequality $|\int f d\mu| \leq \int |f| d\mu$.

For $\bar{I}_f$ we judge analogously.

REMARK 9 I_f (resp. $\bar{I}_f$) is absolutely continuous with respect to m_T (resp. m_S), because $I_1(\tilde{E}) \leq m_T(\tilde{E})$ (resp. $\bar{I}_1(\tilde{E}) \leq m_S(\tilde{E})$).

REFERENCES

1. Zadeh, L.A., Probability Measure of Fuzzy Events, J. Math. Anal. Appl. 23, 1968.

2. Halmos, P.R., Measure Theory, Van Nostrand Reinhold Company, New-York, 1964.

3. Roventa, E., A propos des $\tilde{\mathcal{M}}$-mesures tronçon et support, BUCEFAL (L.S.I. Univ. P. Sabatier, Toulouse) n° 19, 42-48, 1984.

4. Roventa, E., Some Problems Concerning Topological Structures and Measures Theory for Fuzzy Sets and Applications, Ph.D. Thesis, Timisoara University, Romania, 1979. (Romanian)

5. Negoita, C.V., Ralescu, D.A., Simulation, Knowledge-based Computing, and Fuzzy Statistics, Van Nostrand Reinhold Company, New York, 1987.

6. Puri, M.L., Ralescu, D.A., Integration on Fuzzy Sets, Advances in Applied Mathematics 3, 430-434, 1982.

7. Dubois, D., Prade, H., Evaluation of Fuzzy Sets and Fuzzy Integration: A Synthetic Discussion; Second Joint IFSA - EC and EURO - WG Workshop, Vienna, April 6-8, 131-135, 1988.

8. Dubois, D., Prade, H., Theorie des possibilités, Masson, 1985.

9. Sugeno, M., Theory of Fuzzy Integrals and its Applications, PhD. Thesis, Tokyo Institute of Technology, 1974.

10. Yager, R.R., A Note on Probabilities of Fuzzy Events, Information Sciences, 18, 113-129, 1979.