FUZZY INTEGRAL ON FUZZY SET

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The fuzzy integral on fuzzy set is introduced and some of its properties are presented and a series of convergence theorems are given.

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1. Introduction

In Section 2 of this paper, the concepts of stratiformization limit of fuzzy subsets of $\overline{R}$ and extended fuzzy number will be introduced, which will play an important role in the discussion on the newly-defined fuzzy integral. In Section 3, we will introduce an integral, which assumes values in the class of fuzzy subsets of $\overline{R}$, defined on a fuzzy set and give some of its elementary properties. In Section 4, we will prove a series of convergence theorems for the sequence of fuzzy integrals.

Throughout the paper, assume that X is a non-empty set and $\mathfrak{F}$ is a σ-algebra of subsets of X, $\overline{R}=[-\infty, +\infty]$, and use $\mathcal{F}(X)$ to denote the class of all fuzzy subsets of X. Let μ be the fuzzy measure[4] defined on $\mathfrak{F}$, T be index set. Furthermore, we make the following conventions: $\sup\{i: i\in\emptyset\}=0$, $\infty-\infty=0$, $0\cdot\infty=0$, and $\Sigma_T\cdot=0$ when T is empty.

2. Stratiformization limit and extended fuzzy number

For convenience' sake, write $h_\lambda=\{x: h(x)\geq\lambda\}$, where $\lambda\in(0,1]$, $h\in\mathscr{F}(X)$.

Definition 2.1. Let h_1, $h_2\in\mathscr{F}(\bar{R})$, if $(h_i)_\lambda\neq\emptyset$, $\lambda\in(0,1]$, $i=1,2$, write $a_i^\lambda=\inf(h_i)_\lambda$; $b_i^\lambda=\sup(h_i)_\lambda$. We call $h_1\preccurlyeq h_2$(or $h_2\succcurlyeq h_1$) iff there holds that $(h_1)_\lambda\neq\emptyset$ implies $(h_2)_\lambda\neq\emptyset$ and $a_1^\lambda\leq a_2^\lambda$, $b_1^\lambda\leq b_2^\lambda$ for any $\lambda\in(0,1]$.

It is easy to see that $\preccurlyeq$ is a pre-order on $\mathscr{F}(\bar{R})$. We call h_1 and h_2 equivalent if both $h_1\preccurlyeq h_2$ and $h_2\preccurlyeq h_1$ are valid, write as $h_1\sim h_2$. Obviously, the relation $\sim$ is an equivalent relation on $\mathscr{F}(\bar{R})$.

Let $h_n\in\mathscr{F}(\bar{R})$, $n=1,2,\ldots$. If $(h_n)_\lambda\neq\emptyset$, $\lambda\in(0,1]$, write $a_n^\lambda=\inf(h_n)_\lambda$; $b_n^\lambda=\sup(h_n)_\lambda$. For all $\lambda\in[0,1]$, if $\{h_n\}$ satisfies the following condition (I) or (II):

(I) there exists n_λ such that $(h_n)_\lambda\neq\emptyset$ as $n\geq n_\lambda$, and both $\lim_{n\to\infty} a_n^\lambda$ and $\lim_{n\to\infty} b_n^\lambda$ exist.

(II) there exists n_λ such that $(h_n)_\lambda\neq\emptyset$ as $n\geq n_\lambda$.

Then put

$$H_1(\lambda)=\begin{cases}[a^\lambda,b^\lambda] & \text{if there exists } n_\lambda \text{ such that } (h_n)_\lambda\neq\emptyset \text{ for every } n\geq n_\lambda \text{ and } \lim_{n\to\infty} a_n^\lambda=a^\lambda,\ \lim_{n\to\infty} b_n^\lambda=b^\lambda,\\ \emptyset & \text{if there exists } n_\lambda \text{ such that } (h_n)_\lambda=\emptyset \text{ for every } n\geq n_\lambda.\end{cases}$$

Definition 2.2. Let $h_n\in\mathscr{F}(\bar{R})$, $n=1,2,\ldots$, for every $\lambda\in(0,1]$, $\{h_n\}$ satisfies condition (I) or (II). The fuzzy set determined by nest of sets $\{H_1(\lambda): \lambda\in[0,1]\}$ is called the stratiformization limit of $\{h_n\}$, denoted by $(s)\lim_{n\to\infty} h_n$, where no danger of confusion exists, we simply write h_1, $h_1=(s)\lim_{n\to\infty} h_n$, that is, the membership function $m_{h_1}(x)$ of h_1 is defined as

$$m_{h_1}(x)=\bigvee_{\lambda\in[0,1]}\{\lambda\wedge\chi_{H_1(\lambda)}(x)\}.$$

Definition 2.3. $\underset{\sim}{A}\in\mathscr{F}(\bar{R})$ is called a extended fuzzy number, if $m_{\underset{\sim}{A}}$ satisfies the following conditions:

(1) there exists $x_0\in\bar{R}$ such that $m_{\underset{\sim}{A}}(x_0)=1$,

(2) A_λ is a closed interval for any $\lambda\in[0,1]$.

The set of all extended fuzzy numbers is denoted by $\mathscr{F}^*(\bar{R})$.

Definition 2.4. Let $\underset{\sim}{A}\in\mathscr{F}^*(\bar{R})$. $\underset{\sim}{A}$ is called a nonnegative extended fuzzy number if $m_{\underset{\sim}{A}}(x)=0$ for all $x\in[-\infty,0)$.

The set of all nonnegative extended fuzzy numbers is denoted by $\mathscr{F}^*(R^+)$.

The following propositions can be easily obtained from the above definitions.

Proposition 2.1. The relation $\preccurlyeq$ restricted to $\mathscr{F}^*(\bar{R})$ is a partial order.

Proposition 2.2. Let $h_n\in\mathscr{F}(\bar{R})$, $n=1,2,\ldots$, and $m_{h_n}(x)=\begin{cases}1 & x=a_n,\\ 0 & x\neq a_n.\end{cases}$ If $\lim\limits_{n\to\infty} a_n$ exists, then

$$m_{h_1}(x)=\begin{cases}1 & x=\lim\limits_{n\to\infty} a_n,\\ 0 & x\neq\lim\limits_{n\to\infty} a_n.\end{cases}$$

This shows that the stratiformization limit is a generalization of classical limit.

Proposition 2.3. Let $h_n\in\mathscr{F}^*(\bar{R})$, $n=1,2,\ldots$, and $(h_n)_\alpha=[a_n^\alpha, b_n^\alpha]$, $\alpha\in[0,1]$. If both $\lim\limits_{n\to\infty} a_n^\alpha$ and $\lim\limits_{n\to\infty} b_n^\alpha$ exist for all $\alpha\in[0,1]$, then $(s)\lim\limits_{n\to\infty} h_n\in\mathscr{F}^*(\bar{R})$, and for any $\lambda\in(0,1]$,

$$[(s)\lim_{n\to\infty} h_n]_\lambda=[\lim_{\alpha\uparrow\lambda}\lim_{n\to\infty} a_n^\alpha,\ \lim_{\alpha\uparrow\lambda}\lim_{n\to\infty} b_n^\alpha].$$

Proof. $$[(s)\lim_{n\to\infty} h_n]_\lambda=\bigcap_{\alpha<\lambda}[\lim_{n\to\infty} a_n^\alpha,\ \lim_{n\to\infty} b_n^\alpha]$$

$$=[\lim_{\alpha\uparrow\lambda}\lim_{n\to\infty} a_n^\alpha,\ \lim_{\alpha\uparrow\lambda}\lim_{n\to\infty} b_n^\alpha].$$

Proposition 2.4. Let h_n, $l_n \in \mathscr{F}(\bar{R})$ and $h_n \preccurlyeq l_n$ for every n. If both $(s)\lim_{n\to\infty}(h_n)$ and $(s)\lim_{n\to\infty}(l_n)$ exist, then

$$(s)\lim_{n\to\infty} h_n \preccurlyeq (s)\lim_{n\to\infty} l_n.$$

3. F-integral on fuzzy set

Let $M=\{f$: f is a nonnegative, extended real-valued measurable function on $X\}$, $\mathscr{A}=\{\underset{\sim}{A}: \underset{\sim}{A}\in\mathscr{F}(X)$ and $m_{\underset{\sim}{A}}\in M\}$.

Definition 3.1. Let $f\in M$, $\underset{\sim}{A}\in\mathscr{A}$. The F-integral of f over $\underset{\sim}{A}$ is defined as a fuzzy subset of $\bar{R}$, denoted by $(F)\int_{\underset{\sim}{A}} f\, d\mu$, whose membership function is defined as

$$\sup\{\lambda:\ \int_{A_1} f\, d\mu \le x \le \int_{A_\lambda} f\, d\mu\},$$

where $\int_{A_\lambda} f\, d\mu$ is in sense of Sugeno, that is, $\int_{A_\lambda} f\, d\mu = \sup_{\alpha\in[0,\infty]}\{\alpha\wedge\mu(A_\lambda\cap F_\alpha)\}$, where $F_\alpha=\{x:\ f(x)\ge\alpha\}$.

Proposition 3.1. Let $f\in M$, $\underset{\sim}{A}\in\mathscr{A}$, then the F-integral of f over $\underset{\sim}{A}$ can be expressed as

$$(F)\int_{\underset{\sim}{A}} f\, d\mu = \bigcup_{\lambda\in[0,1]} \lambda H(\lambda),$$

where $H(\lambda)=[\int_{A_1} f\, d\mu,\ \int_{A_\lambda} f\, d\mu]$, $\bigcup_{\lambda\in[0,1]}\lambda H(\lambda)$ is a fuzzy set whose membership function is defined as: $\sup\{\lambda\cdot\chi_{H(\lambda)}(x):\ \lambda\in[0,1]\}$.

Proposition 3.2. Let $\underset{\sim}{A}=A\in\mathscr{B}$, write $\int_A f\, d\mu=y$ and $(F)\int_{\underset{\sim}{A}} f\, d\mu=\underset{\sim}{C}$, then

$$m_{\underset{\sim}{C}}(x)=\begin{cases}1 & x=y,\\ 0 & x\neq y.\end{cases}$$

This indicates that the above-introduced F-integral is a generalization of that of Sugeno's.

Proposition 3.3. Whenever f=g a.e. and $m_{\underset{\sim}{A}}=m_{\underset{\sim}{B}}$ a.e., $f,g\in M$, $\underset{\sim}{A},\underset{\sim}{B}\in\mathscr{A}$, then $(F)\int_{\underset{\sim}{A}} f\, d\mu=(F)\int_{\underset{\sim}{B}} g\, d\mu$, iff μ is null-additive[4].

Proposition 3.4. Let $f,g\in M$, $\underset{\sim}{A},\underset{\sim}{B}\in\mathscr{A}$. If $f\le g$ and $\underset{\sim}{A}\subseteq\underset{\sim}{B}$, then

$$(F)\int_{\underset{\sim}{A}} f\, d\mu \preccurlyeq (F)\int_{\underset{\sim}{B}} g\, d\mu.$$

Proposition 3.5. Let $f\in M$, $\underset{\sim}{A}\in\mathscr{A}$, and write $\mathring{A}=\{x:\ m_{\underset{\sim}{A}}(x)\ge 0\}$, then

$(F)\int_{\underset{\sim}{A}} f\ d\mu = \underset{\sim}{0} \Longleftrightarrow \mu(\{f>0\}\cap \mathring{A})=0$, where $\underset{\sim}{0}\in\mathcal{A}$ and $m_{\underset{\sim}{0}}(x)=\begin{cases}1 & x=0,\\ 0 & x\neq 0.\end{cases}$

Theorem 3.1. Let $f\in M$, $\underset{\sim}{A}\in\mathcal{A}$, then $\underset{\sim}{C}=(F)\int_{\underset{\sim}{A}} f\ d\mu \in \mathcal{F}(\bar{R})$ and for any $\lambda\in(0,1]$,

$$C_\lambda = [\int_{A_1} f\ d\mu,\ \lim_{\alpha\uparrow\lambda}\int_{A_\alpha} f\ d\mu].$$

Theorem 3.2. Let $f\in M$, $\underset{\sim}{A}\in\mathcal{A}$. If $\{f,\underset{\sim}{A}\}$ satisfies the following condition:

(*) for every $\lambda\in(0,1]$, there exists $\lambda'<\lambda$ such that

$$\mu(\{f\cdot\chi_{A_{\lambda'}} > \int_{A_\lambda} f\ d\mu\})<\infty,$$

then for any $\lambda\in(0,1]$

$$[(F)\int_{\underset{\sim}{A}} f\ d\mu]_\lambda = [\int_{A_1} f\ d\mu,\ \int_{A_\lambda} f\ d\mu].$$

Proof. From Theorem 3.1. we have

$$\begin{aligned}(F)\int_{\underset{\sim}{A}} f\ d\mu &= [\int_{A_1} f\ d\mu,\ \lim_{\alpha\uparrow\lambda}\int_{A_\alpha} f\ d\mu|\\ &= [\int_{A_1} f\ d\mu,\ \lim_{\alpha_n\uparrow\lambda}\int_{A_{\alpha_n}} f\ d\mu]\end{aligned}$$

for all $\lambda\in(0,1]$. Since $\bigcap_{n=1}^{\infty} A_{\alpha_n}=A_\lambda$, $f\cdot\chi_{A_{\alpha_n}} \searrow f\cdot\chi_{A_\lambda}$. By using condition (*) and Theorem 14 in [4], we have

$$\int_{A_{\alpha_n}} f\ d\mu = \int f\cdot\chi_{A_{\alpha_n}} d\mu \longrightarrow \int f\cdot\chi_{A_\lambda} d\mu = \int_{A_\lambda} f\ d\mu,$$

and consequently,

$$[(F)\int_{\underset{\sim}{A}} f\ d\mu]_\lambda = [\int_{A_1} f\ d\mu,\ \int_{A_\lambda} f\ d\mu].$$

Corollary 3.1. Let $f\in M$, $\underset{\sim}{A}\in\mathcal{A}$. If $\{f,\underset{\sim}{A}\}$ satisfies: $\mu(\{f>\int_{A_1} f\ d\mu\})<\infty$, then for any $\lambda\in(0,1]$,

$$[(F)\int_{\underset{\sim}{A}} f\ d\mu]_\lambda = [\int_{A_1} f\ d\mu,\ \int_{A_\lambda} f\ d\mu].$$

Corollary 3.2. Let $\mu(X)<\infty$, then for any $f\in M$, $\underset{\sim}{A}\in\mathcal{A}$, and $\lambda\in(0,1]$,

$$[(F)\int_{\underset{\sim}{A}} f\ d\mu]_\lambda = [\int_{A_1} f\ d\mu,\ \int_{A_\lambda} f\ d\mu].$$

4. The convergence theorems

Theorem 4.1. Let $\underset{\sim}{A}\in\mathcal{A}$, $f, f_n\in M$, $n=1,2,\ldots$. If $\{f_n\}$ converges uniformly to f on $\mathring{A}=\{x:\ m_{\underset{\sim}{A}}(x)>0\}$, then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu = (F)\int_{\underset{\sim}{A}} f\ d\mu.$$

Proof. By using Theorem 3.1 we have

$$[(F)\int_{\underset{\sim}{A}} f_n d\mu]_\lambda = [\int_{A_1} f_n d\mu,\ \lim_{\alpha\uparrow\lambda}\int_{A_\alpha} f_n d\mu]$$

and

$$[(F)\int_{\underset{\sim}{A}} f\ d\mu]_\lambda = [\int_{A_1} f\ d\mu,\ \lim_{\alpha\uparrow\lambda}\int_{A_\alpha} f\ d\mu]$$

hold for every n and all $\lambda\in(0,1]$. For any $\varepsilon>0$, since $\{f_n\}$ converges uniformly to f on $\mathring{A}$, there exists N such that, as $n\geqslant N$, $|f_n(x)-f(x)|\leqslant\varepsilon$ for all $x\in\mathring{A}$. By using Theorem 9 in [4], as $n\geqslant N$,

$$\left|\int_{A_\alpha} f_n d\mu - \int_{A_\alpha} f\ d\mu\right| \leqslant \varepsilon$$

holds for all $\alpha\in(0,1]$, it follows that

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu = (F)\int_{\underset{\sim}{A}} f\ d\mu.$$

Theorem 4.2. Let $f_n\in M$, $n=1,2,\ldots$, $\underset{\sim}{A}\in\mathcal{A}$. If $f_n\nearrow f$ and there exists N such that, as $n\geqslant N$, $\{f_n,\underset{\sim}{A}\}$ and $\{f,\underset{\sim}{A}\}$ satisfy condition (*), that is, for any $\lambda\in(0,1]$, there exist $\lambda_n<\lambda$ and $\lambda'<\lambda$ such that

$$\mu(\{f_n\cdot\chi_{A_{\lambda_n}} > \int_{A_\lambda} f_n d\mu\})<\infty,$$
$$\mu(\{f\cdot\chi_{A_{\lambda'}} > \int_{A_\lambda} f\ d\mu\})<\infty,$$

then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu = (F)\int_{\underset{\sim}{A}} f\ d\mu.$$

Corollary 4.1. Let $f_n\in M$, $n=1,2,\ldots$, $\underset{\sim}{A}\in\mathcal{A}$. If $f_n\nearrow f$ and there exists N such that $\mu(\{f>\int_{A_1} f_N d\mu\})<\infty$, then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu = (F)\int_{\underset{\sim}{A}} f\ d\mu.$$

Theorem 4.3. Let $f_n\in M$, $n=1,2,\ldots$, $\underset{\sim}{A}\in\mathcal{A}$. If $f_n\searrow f$ and there exists N such that $\mu(\{f_N>\int_{A_1} f\ d\mu\})<\infty$, then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu = (F)\int_{\underset{\sim}{A}} f\ d\mu.$$

Corollary 4.2. Let $f_n\in M$, $n=1,2,\ldots$, $\underset{\sim}{A}\in\mathcal{A}$, and μ is null-additive

(1) If $f_n\nearrow f$ a.e. and there exists N such that $\mu(\{f>\int_{A_1} f_N d\mu\})<\infty$, then $(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu = (F)\int_{\underset{\sim}{A}} f\ d\mu$.

(2) If $f_n\searrow f$ a.e. and there exists N such that $\mu(\{f_N>\int_{A_1} f\ d\mu\})<\infty$, then $(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu = (F)\int_{\underset{\sim}{A}} f\ d\mu$.

Theorem 4.4. Let $f_n\in M$, $n=1,2,\ldots$, $\underset{\sim}{A}\in\mathcal{A}$. If $\{f_n\}$ converges to f on

X and there exists N such that:

(1) $\mu(\{f>\int_{A_1}\inf_{n\geq N} f_n$

(2) $\mu(\{\sup_{n\geq N} f_n>\int_{A_1} f\ d\mu\})<\infty$,

then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu=(F)\int_{\underset{\sim}{A}} f\ d\mu.$$

Proof. Put $g_n=\inf_{m\geq n} f_m$; $h_n=\sup_{m\geq n} f_m$, then

$$g_n\leq f_n\leq h_n$$

for every n, and $g_n \nearrow f$, $h_n \searrow f$.

From conditions (1) and (2), we have

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} g_n d\mu=(F)\int_{\underset{\sim}{A}} f\ d\mu,$$

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} h_n d\mu=(F)\int_{\underset{\sim}{A}} f\ d\mu.$$

Furthermore, by using Proposition 3.4,

$$(F)\int_{\underset{\sim}{A}} g_n d\mu \preccurlyeq (F)\int_{\underset{\sim}{A}} h_n d\mu \preccurlyeq (F)\int_{\underset{\sim}{A}} h_n d\mu$$

holds for every n. According to Proposition 2.4, we have

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} g_n d\mu \preccurlyeq (s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu \preccurlyeq (s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} h_n d\mu,$$

and therefore

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu=(F)\int_{\underset{\sim}{A}} f\ d\mu.$$

Corollary 4.3. Let $f_n\in M$, $n=1,2,\ldots$, $\underset{\sim}{A}\in\mathcal{A}$ and $\mu(\mathring{A})<\infty$. If $\{f_n\}$ converges to f on X, then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu=(F)\int_{\underset{\sim}{A}} f\ d\mu.$$

Corollary 4.4. Let $f_n, f\in M$, $\underset{\sim}{A}\in\mathcal{A}$ and $\mu(\mathring{A})<\infty$. If $f_n\to f$ a.e. and μ is null-additive, then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu=(F)\int_{\underset{\sim}{A}} f\ d\mu.$$

Theorem 4.5. Let $\mu(X)<\infty$. Whenever $\{f_n\}$, $\{f_n\}\subseteq M$, converges in measure to an a.e. finite measurable function f, then $(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu=(F)\int_{\underset{\sim}{A}} f\ d\mu$ holds for all $\underset{\sim}{A}\in\mathcal{A}$, iff μ is autocontinuous.

Proof. Necessity. Since $\mathcal{B}\subseteq\mathcal{A}$, the necessity is obtained by using Theorem 16 in[4].

Sufficiency. For any $\lambda\in(0,1]$, it is easy to see that

$$[(F)\int_{\underset{\sim}{A}} f\, d\mu]_\lambda=[\int_{A_1} f\, d\mu,\ \int_{A_\lambda} f\, d\mu)$$

and

$$[(F)\int_{\underset{\sim}{A}} f_n d\mu]_\lambda=[\int_{A_1} f_n d\mu,\ \int_{A_\lambda} f_n d\mu)$$

hold for every n. According to Theorem 16 in [4], we have

$$\lim_{n\to\infty}\int_{A_\lambda} f_n d\mu=(F)\int_{A_\lambda} f\, d\mu,$$

it results that

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu=(F)\int_{\underset{\sim}{A}} f\, d\mu.$$

Theorem 4.6. Let $f_n\in M$, $n=1,2,\ldots$, $\underset{\sim}{A}\in\mathcal{A}$, μ is autocontinuous. If $\{f_n\}$ converges in measure to an a.e. finite measurable function f and there exists N such that $\{f,\underset{\sim}{A}\}$ and $\{f_n,\underset{\sim}{A}\}$ satisfy condition (*) for every $n\geq N$, then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu=(F)\int_{\underset{\sim}{A}} f\, d\mu.$$

Corollary 4.5. Let $f_n\in M$, $n=1,2,\ldots$, $\underset{\sim}{A}\in\mathcal{A}$, $\mu(\mathring{A})<\infty$, and μ is autocontinuous. If $\{f_n\}$ converges in measure to an a.e. finite measurable function f, then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}} f_n d\mu=(F)\int_{\underset{\sim}{A}} f\, d\mu.$$

Definition 4.1. Let $f_n\in M$, $n=1,2,\ldots$, $\{f_n\}$ is said to F-mean converges to an a.e. finite measurable function f, if

$$\lim_{n\to\infty}\int|f_n-f|\,d\mu=0.$$

According to Theorem 12 in [4], F-mean convergence is equivalent to convergence in measure. If the condition that $\{f_n\}$ converges in measure is replaced by the condition that $\{f_n\}$ converges in F-mean in Theorem 4.5, Theorem 4.6 and Corollary 4.5, then the corresponding conclusions still hold.

Theorem 4.7. Let $f\in M$, $\underset{\sim}{A},A_n\in\mathcal{A}$, $n=1,2,\ldots$, and $\mu(\mathring{A})<\infty$. if there exists N such that, as $n\geq N$, $\mu(\mathring{A}_n)<\infty$ and $\{\underset{\sim}{A},\underset{\sim}{A}_n\}$ satisfies the following condition:

(**) $\lim_{n\to\infty}((A_n)_\lambda\Delta A_\lambda)=\emptyset$ for any $\lambda\in(0,1]$, then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}_n} f\, d\mu=(F)\int_{\underset{\sim}{A}} f\, d\mu.$$

Theorem 4.8. Let $f\in M$, f be an a.e. finite function, μ is autocontinuous, $\underset{\sim}{A},\underset{\sim}{A}_n\in\mathcal{A}$, $n=1,2,\ldots$, and $\mu(\mathring{A})<\infty$. If there exists N such that, as $n\geq N$, $\mu(\mathring{A}_n)<\infty$ and $\{\underset{\sim}{A},\underset{\sim}{A}_n\}$ satisfies the following condition:

(***) $\lim_{n\to\infty}\mu((A_n)_\lambda \Delta A_\lambda)=0$ for any $\lambda\in(0,1]$,

then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}_n} f\, d\mu=(F)\int_{\underset{\sim}{A}} f\, d\mu.$$

Corollary 4.6. Let f be an a.e. finite measurable function, μ is autocontinuous, $\underset{\sim}{A},\underset{\sim}{A}_n\in\mathcal{A}$, $n=1,2,\ldots$, and $\mu(\mathring{A})<\infty$. If there exists N such that, as $n\geq N$, $\mu(\mathring{A}_n)<\infty$ and $\{\underset{\sim}{A},\underset{\sim}{A}_n\}$ satisfies condition: $\lim_{n\to\infty}\mu(\{x: m_{\underset{\sim}{A}_n}(x)\neq m_{\underset{\sim}{A}}(x)\})=0$, then

$$(s)\lim_{n\to\infty}(F)\int_{\underset{\sim}{A}_n} f\, d\mu=(F)\int_{\underset{\sim}{A}} f\, d\mu.$$

In fact, if $\{\underset{\sim}{A},\underset{\sim}{A}_n\}$ satisfies condition (**), then $\{m_{\underset{\sim}{A}_n}\}$ converges everywhere to $m_{\underset{\sim}{A}}$. Conversly, even if $\{m_{\underset{\sim}{A}_n}\}$ converges uniformly to $m_{\underset{\sim}{A}}$ on X, we cannot conlude that $\{\underset{\sim}{A}_n,\underset{\sim}{A}\}$ satisfies condition (**).

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