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Generalized Convex Fuzzy Sets

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Abstract. In this paper, new classes of generalized convex fuzzy sets are introduced, extending the concepts of convex fuzzy sets, strongly convex fuzzy sets, strictly convex fuzzy sets, and their associate subclasses. Some properties are investigated in this paper.

Key word. Generalized convex fuzzy set, continuous connectedness, convex fuzzy set, level set.

1. Introduction

Convex fuzzy sets were first defined by zadeh in[1]. Some properties were subsequently studies by Brown in[2], weiss in [3], katsaras and liu in [4], lowen in [5] liu in [6]. In this paper we extend the notion of convex fuzzy sets, and present generalized convex fuzzy sets. That is, generalized convex fuzzy sets, generalized strongly convex fuzzy sets and strictly convex fuzzy sets. Some properties of them are discussed in this paper.

2. Extensions of Convex Fuzzy Sets.

Throughout this paper E will denote the n-dimensional Euclidena space R^n. Fuzzy sets and values will be denoted by lower case Greek letters and we shall make no difference between notations for a fuzzy set with a constant value and that value itself.

Definition 2.1. The set $C \subset R^n$ is said to be

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continuous connected, if for every pair of points $x^1 \in \mathcal{C}$, $x^2 \in \mathcal{C}$, there exists a continuous vector valued function H_{x^1,x^2} defined on the unit interval $[0,1] \subset R$ and with values in C such that $H_{x^1,x^2}(0)=x^1$, $H_{x^1,x^2}(1)=x^2$. ......(1)

In the sequel, H_{x^1,x^2} will denote a continuous are connecting x^1 with x^2. Note that every convex set in R^n is continuous connected, since the function $H_{x^1,x^2}(\theta) = (1-\theta)x^1+\theta x^2$ ......(2) is a continuous vector function. In the sense of the above definition. Thus, the concept of continuous connected sets is a generalization of convex sets.

In this work, we shall focus our attention on certain extensions of known families of convex fuzzy sets, strongly convex fuzzy sets, strictly convex fuzzy sets. We first extend the concept of convex fuzzy sets.

Definition 2.2 A fuzzy set λ, defined on R^n, is called generalized convex fuzzy set if, for every $x^1 \in R^n$, $x^2 \in R^n$ such that $\lambda(x^1) \leq \lambda(x^2)$ there exists a continuous vector-valued function $H_{x,x}$ satidfying

$$\lambda(H_{x^1,x^2}(\theta)) \geq \lambda(x^1) \quad ......(3)$$

for $0 \leq \theta \leq 1$.

Convex fuzzy sets are generalized convex fuzzy sets with H_{x^1,x^2} given by (2), it can be shown that λ is generalized convex fuzzy set iff the level sets of λ, given by $L_\alpha(\lambda)=\{x \mid \lambda(x) \geq \alpha\}$ ...... (4) are continuous connected for every $\alpha \in R$.

Following the same idea as above, we introduce additional families of fuzzy sets.

Definition 2.3. A fuzzy set λ, defined on R^n, is called strongly generalized convex fuzzy set if, for every $x^1 \in R^n$, $x^2 \in R^n$, $x^1 \neq x^2$, such athat $\lambda(x^1) \leq \lambda(x^2)$, there exists a continuous vector-valued function H_{x^1, x^2} satisfying $\lambda(H_{x^1, x^2}(\theta)) > \lambda(x^1)$ ...... (5) for $0 < \theta < 1$. Under the same assumptions, λ is called strictly generalized convex fuzzy set, if (5) is satisfied for every $x^1 \in R^n$, $x^2 \in R^n$ such that $\lambda(x^1) < \lambda(x^2)$.

Clearly, strongly and strictly convex fuzzy sets (Ref.1, 7) are strongly and strictly generalized convex fuzzy sets, respectively.

3. Some Properties of Continuous connected Sets.

In this section we give some properties of continuous connected sets.

The following results are easy to establish.

Proposition 3.1 Suppose that X and Y are continuous connected sets of R^n. Then

(i) X+Y is continuous connected set;

(ii) αX is continuous connected set for any real number α;

(iii) X+x is continuous connected set for any $x \in R^n$

Propersition 3.2 Let f be a continuous function from R^m to R^n; then, if $A \subset R^m$ is continuous connected set, so is f(A).

Proof. let $A \subset R^m$ be a continuous connected set. Then, for every pair of points a^1, a^2 in A, there exists a continuous function H_{a^1, a^2} such that $H_{a^1, a^2}(\theta)$ in A. let b^1, b^2 in f(A). There exist a^1, a^2 in A such that

$$f(a^1)=b^1, \quad f(a^2)=b^2$$

Now, $(f \circ H_{a^1, a^2})(0) = f(H_{a,a}(0)) = f(a^1) = b^1$,

$(f \circ H_{a^1, a^2})(1) = f(H_{a,a}(1)) = f(a^2) = b^2$.

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Moreover, foH_{a^1,a^2} is continuous and

$$foH_{a^1,a^2}(\theta) \subset f(A) \quad \text{for} \quad 0 \leq \theta \leq 1$$

Proposition 3.3 let C and D be continuous connected subsets of R^m and R^p, respectively. Then

$$C+D = \{(c,d): c \in C,\ d \in D\}$$

is also continuous connected set of R^{m+p}.

Proof. Suppose that (c^1,d^1), $(c^2,d^2) \in C+D$. Then, there exist continuous H_{c^1,c^2}, H_{d^1,d^2} such that

$$H_{c^1,c^2}(0)=c^1, \qquad H_{c^1,c^2}(1)=c^2$$
$$H_{d^1,d^2}(0)=d^1, \qquad H_{d^1,d^2}(1)=d^2$$
$$H_{c^1,c^2}(\theta) \subset C, \qquad H_{d^1,d^2}(\theta) \subset D \text{ for } \quad 0 \leq \theta \leq 1.$$

Now, $(H_{c^1,c^2}, H_{d^1,d^2})$ is the desired continuous function for the pair $((c^1,d^1), (c^2,d^2))$.

It is obvious that the above result can be established for the direct sum of any finite number of subsets.

4. Some Properties of Gereralized Convex Fuzzy Sets

In this section, we shall deal with some properties of the classes of sets that were introduced in the preceding section. Here many results are generalization of corresponding results for convex fuzzy sets due to yang (Ref.7)

We now prove a necessary and sufficient condition for a strictly generalized convex fuzzy set to be gereralized convex fuzzy set. let

$$G(\lambda) = \{\alpha \in R: L_\alpha(\lambda) \neq \emptyset\}$$

and let

$$S = \begin{cases} L_\alpha \sup(\lambda), & \text{if } G(\lambda) \text{ is bounded above.} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Where $\alpha_{sup} = \sup\{\alpha \in R;\ \alpha \in G(\lambda)\}$

Then, we have the following theorem.

Theorem 4.1 let λ be a strictly generalized convex fuzzy set defined on R^n. Then, λ is generalized convex fuzzy set iff $\bar{S}_\lambda$ is continuous connected set.

Proof. Dy definition, necessary is obviously. Assume now that $\bar{S}_\lambda$ is continuous connected set. We only have to show that $x^1 \in R^n$, $x^2 \in R^n$, and $\lambda(x^1) = \lambda(x^2)$ imply the existence of a continuous function satisfy (3) of Definition 2.2, since the case $\lambda(x^1) < \lambda(x^2)$ is covered by Definition 2.3. If $\bar{S}_\lambda \neq \phi$ and $x^1 \in \bar{S}_\lambda$, $x^2 \in \bar{S}_\lambda$, then we are done, by hypothese. Otherwise, there exists a point $\bar{x} \in R^n$ satisfying

$$\lambda(\bar{x}) > \lambda(x^1) = \lambda(x^2).$$

Since λ is strictly generalized convex fuzzy set, we have $H_{\bar{x},x^2}$ and $H_{\bar{x},x^1}$ such that

$$\lambda(H_{\bar{x},x^1}(\theta)) > \lambda(x^1) \qquad 0 < \theta < 1$$

$$\lambda(H_{\bar{x},x^2}(\theta)) > (x^2) \qquad 0 < \theta < 1$$

Take

$$H_{x^1,x^2}(\theta) = \begin{cases} H_{\bar{x},x^1}(1-2\theta) & 0 \leq \theta \leq \frac{1}{2} \\ H_{\bar{x},x^2}(2\theta-1) & \frac{1}{2} \leq \theta \leq 1 \end{cases} \qquad \ldots\ldots \quad (6)$$

Then, the continuous function given by (6) satisfies (3). of Definition 2.2.

Theorem 4.2. let λ be a strictly generalized convex fuzzy set defined on R^n, and let $x^* \in R^n$ be a local maximum of λ. Then, x^* is a global maximum of λ over R^n.

Proof. Suppose that x^* is a local maximum; i.e., there exists an open ball $N_\delta(x^*)$ such that, for every $x \in N_\delta(x^*)$, we have $\lambda(x^*) \geqslant \lambda(x)$ ....(7)

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suppose now that there exists a point $\bar{x} \in R^n$ satisfying

$$\lambda(\bar{x}) > \lambda(x^*)$$

Then, since λ be strictly generalized convex fuzzy set, we have a continuous function $H_{\bar{x},x^*}$ satisfying

$$\lambda(H_{\bar{x},x^*}(\theta)) > \lambda(x^*)$$

for every $0 < \theta < 1$. It follows from the continuity of $H_{\bar{x},x^*}(\theta) \in N(x^*)$ as θ approaches 1, conrtadicting (7).

For strongly generalized convex fuzzy set, we have the folloeing result, its proof is Similar to theorem 4.2.

Theorem 4.3 Let λ be a stongly generalized convex f fuzzy set defined on R^n, and let x^* be a local maximum of λ. Then, x^* is a unique global maximum of λ over R^n.

Now, let $L^{\circ}_{\alpha}(\lambda)=\{x \in R^n: \lambda(x)>\alpha\}$ denote the strict level set of λ defined on R^n at $\alpha \in R$.

Then, we have the following definition.

Definition 4.1. let λ be a fuzzy set defined on R^n. Then, λ is said to be with property (A) if, for every $\alpha \in R$ such that $L^{\circ}_{\alpha}(\lambda)$ is continuous connected set, it follow that $L^{\circ}_{\alpha}(\lambda) \cup \{\hat{x}\}$ is continuous connected set for every $\hat{x} \in cl L^{\circ}_{\alpha}(\lambda)$, where cl indicates the closure operation of sets in R^n.

In the sequel, we shall mainly deal with fuzzy set with property (A). Our result deal with conditions under which a generalized convex fuzzy set is also strictly generalized convex fuzzy set. First, we have the following lemma.

Lemma 4.1 let λ be fuzzy set defined on R^n with property (A), and assume that $L^{\circ}_{\alpha}(\lambda)$ is continuous connected set for every $\alpha \in R$. Suppose that $x^1 \in R^n$, $x^2 \in R^n$

$\lambda(x^1) < \lambda(x^2)$, and x^2 is not local maximum of λ. Then, there exists a continuous function H_{x^1,x^2} satisfying

$$\lambda(H_{x^1,x^2}(\theta)) > \lambda(x^1) \quad \ldots\ldots\ldots(8)$$

for every $0 < \theta < 1$

Proof. let $\hat{\alpha} = \lambda(x^1)$. Then, $x^2 \in L^{\circ}_{\hat{\alpha}}(\lambda)$, and we can find a sequence $\{x^i\} \subset L^{\circ}_{\hat{\alpha}}(\lambda)$, converging to x^1 such that $x^i \in N_{\delta_i}(x^1)$, where $N_{\delta_i}(x^1) = \{x \mid \|x - x^1\| < \delta_i\}$ and $\delta_i = 1/i$, $i=2,3,\ldots\ldots$

Thus, x^1 is a cluster point of $L^{\circ}_{\hat{\alpha}}(\lambda)$. Since λ have property (A), the set $L^{\circ}_{\hat{\alpha}}(\lambda) \cup \{x^1\}$ is continuous connected set, and there exists a continuous function H_{x^1,x^2} contained in $L^{\circ}_{\hat{\alpha}}(\lambda)$ for $0<\theta<1$.

The following lemma will be useful in the sequel.

Lemma 4.2 let λ be a generalized convex fuzzy set defined on R^n. Then, $L^{\circ}_{\alpha}(\lambda)$ is continuous connected set for every $\alpha \in R$.

Proof. Let λ be generalized convex fuzzy set defined on R^n. Then, the sets $L_{\alpha}(\lambda)$ are continuous connected sets for every $\alpha \in R$. Let $\bar{\alpha} \in R$ be such that $L^{\circ}_{\bar{\alpha}}(\lambda)$ is not empty. Then, for every $x^1 \in L^{\circ}_{\bar{\alpha}}(\lambda)$, $x^2 \in L^{\circ}_{\bar{\alpha}}(\lambda)$, we have $\lambda(x^1) > \bar{\alpha}$, $\lambda(x^2) > \bar{\alpha}$. Assume that $\lambda(x^1) \leq \lambda(x^2)$. Since λ is generalized convex fuzzy set, there exists a continuous function H_{x^1,x^2} satisfying $\lambda(H_{x^1,x^2}(\theta)) \geq \lambda(x^1) > \bar{\alpha}$ for every $0 \leq \theta \leq 1$. Hence, H_{x^1,x^2} is in $L^{\circ}_{\bar{\alpha}}(\lambda)$ and $L^{\circ}_{\bar{\alpha}}(\lambda)$ is continuous connected set.

Now, we can state and prove the follow theorem.

Theorem 4.4 let λ be generalized convex fuzzy set defined on R^n with property (A). If every local maximum of λ is a global one over R^n, then λ is strictly generalized convex fuzzy set.

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Proof. Suppose that $x^1 \in R^n, x^2 \in R^n$, and $\lambda(x^1) < \lambda(x^2)$, it follows from the hypothness that x^1 is not a local maximum of λ. By lemma 4.2, $L^{\circ}_{\alpha}(\lambda)$ is continuous connected set for every $\alpha \in R$. Taking $\alpha = \lambda(x^1)$ we conclude (by lemma 4.1) that there exist a continuous function H_{x^1, x^2} satisfying (8), for every $0 \leq \theta \leq 1$. Hence λ is a strictly generalized convex fuzzy set.

Generalized convex fuzzy set have been defined here; Some properties have been discussed. We consider that a there are many areas of this paper in which further work is needed.

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