t-NORMS AND GENERALIZED COPULAS*

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Abstract The purpose of this paper is to generalize the notion of 2-copulas with the aid of an Archimedean t-norm. We give necessary and sufficient conditions for a t-norm to be a generalized copula.

1. Introduction

The concept of a copula was introduced by Sklar in [3].

Definition 1. A continuous function $C : [0,1]\times[0,1] \to [0,1]$ is said to be a *copula* iff the following conditions hold:

(a) $C(0,0) = 0$, $C(a,1) = C(1,a) = a \quad \forall\, a \in [0,1]$

(b) $a \leqq c,\ b \leqq d \Rightarrow C(a,b) \leqq C(c,d)$

(c) $a \leqq c,\ b \leqq d \Rightarrow C(a,d) + C(c,b) \leqq C(a,b) + C(c,d)$.

We remark that the definitions and results were stated for the n-dimensional case in [3].

Assume now that T is any *Archimedean t-norm* , i.e., a function defined on the unit square with the following properties: T is commutative, associative, non-decreasing, continuous and $T(a,1) = a \quad \forall\, a \in [0,1]$, $T(a,a) < a$ $\forall\, a \in (0,1)$ (see e.g. [4] for details). Moreover, let S be any *Archimedean t-conorm*. Define the following operations in $[0,1]$:

$$a\ \alpha_T\ b = \sup\ \{\ x\ ;\ T(a,x) \leqq b\ \},$$
$$a\ \omega_S\ b = \inf\ \{\ x\ ;\ b \leqq S(a,x)\ \}.$$

By the help of these operations we can formulate the definition of a T-copula.

* This work has been partially supported by OTKA-27-5-606.

2. T - copulas

Suppose that T is a given Archimedean t-norm.

Definition 2. A continuous function $C : [0,1]\times[0,1]\to[0,1]$ is called *T-copula* if it satisfies the conditions (a) and (b) and

(c') $C(c,b)\ \alpha_T\ C(a,b) \geqq C(c,d)\ \alpha_T\ C(a,d)$ whenever $a \leqq c,\ b \leqq d$.

One can readily verify that if $T(a,b) = \max(a + b - 1, 0)$ then (c') is the same as (c), i.e., Definition 2 is indeed a generalization of Definition 1.

Theorem 1. Assume that C is a T-copula. Then

$$T(a,b) \leqq C(a,b) \leqq \min(a,b) \quad \forall\ a,b \in [0,1].$$

Proof. In virtue of (a) and (b) $C(a,b) \leqq C(a,1) = a$ and $C(a,b) \leqq C(1,b) = b$, i.e., $C(a,b) \leqq \min(a,b)$.
On the other hand, let $c = d = 1$ in (c'). Then
$C(1,b)\ \alpha_T\ C(a,b) \geqq C(1,1)\ \alpha_T\ C(a,1)$, i.e., $b\ \alpha_T\ C(a,b) \geqq 1\ \alpha_T\ a$.
This is true iff $C(a,b) \geqq T(1\ \alpha_T\ a,b)$. But $1\ \alpha_T\ a = a$.
Hence we get the assertion.

Our second proposition gives necessary and sufficient condition for a t-norm to be a T-copula.

Theorem 2. A continuous t-norm T_1 is a T-copula if and only if

(1) $T_1(c,b)\ \alpha_T\ T_1(a,b) \geqq c\ \alpha_T\ a$ for $c \geqq a$ and $b \in [0,1]$.

Proof. If T_1 is a T-copula then the inequality (c') is valid with $C = T_1$. Let $d = 1$ in (c'). Then we get the first part of our assertion.
Conversely, assume that $a \leqq c$, $b \leqq d$. Then there exists an $e = d\ \alpha_{T_1}\ b$ such that $T_1(e,d) = b$. Hence we get that

$T_1(c,b)\ \alpha_T\ T_1(a,b) = T_1(c,T_1(e,d))\ \alpha_T\ T_1(a,T_1(e,d)) =$
$= T_1(T_1(c,d),e)\ \alpha_T\ T_1(T_1(a,d),e) \geqq T_1(c,d)\ \alpha_T\ T_1(a,d).$
This completes the proof.

We prove in the next theorem that for any Archimedean t-norm T the family of T-copulas is non-empty.

Theorem 3. The t-norms T and 'min' are T-copulas.

Proof.
i) It is easy to verify that
$T(c,b)\ \alpha_T\ T(a,b) = c\ \alpha_T\ [b\ \alpha_T\ T(a,b)]$ and $b\ \alpha_T\ T(a,b) \geqq a$.
Hence, $T(c,b)\ \alpha_T\ T(a,b) \geqq c\ \alpha_T\ a$, i.e., T is a t-copula.
ii) If a and c are given then there are three cases:
$a \leqq c \leqq b$; $a \leqq b \leqq c$; $b \leqq a \leqq c$. One can easily see that (1) holds for 'min' in all three cases, i.e.,'min' is a T-copula.

Assume now that the Archimedean t-norm T is non-strict with normed generator f and let $n_T(a) = a\ \alpha_T\ 0$, $S(a,b) = n_T[T(n_T(a),n_T(b))]$.Weber proved in [4] that in this case S is a non-strict Archimedean t-conorm with normed generator $g(x) = 1 - f(x)$. S is called the complementary t-conorm to T. Finally, let $f^{(-1)}$ be the pseudo-inverse of f.

Theorem 4. Let T be a given Archimedean t-norm and let S be the complementary t-conorm to T with normed generator g. Then an Archimedean t-norm T_1 with generator f_1 is a T-copula if and only if the function $g \circ f_1^{(-1)}$ is convex.

Proof. The proof is carried out analogously to the proofs of Theorem 9 in [1] and Theorem 2 in [2].

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