FURTHER DISCUSSIONS ON FUZZY PAN-INTEGRAL

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ABSTRACT

In paper [1], we gave the concept of the fuzzy pan-integral, discussed its elementary properties and proved some convergence theorems of the sequence of fuzzy pan-integrals. In this paper, the properties of the fuzzy pan-integral are further discussed under the fuzzy pan-additive condition. Several convergence theorems of the sequence of fuzzy pan-integrals are given.

Keywords: Fuzzy Pan-integral, Fuzzy Pan-space, Fuzzy Pan-additivity.

§1 PREPARATION KNOWLEDGE

In this paper, we shall always assume that $(X,\underset{\sim}{\mathcal{F}})$ is a fuzzy measurable space; $(X,\underset{\sim}{\mathcal{F}},\underset{\sim}{\mu})$ is a fuzzy measure space; $(\overline{R}_+,\oplus,\odot)$ is an ordered commutative semi-ring with respect to "$\oplus$" and "$\odot$" on $\overline{R}_+=[0,\infty]$; $(X,\underset{\sim}{\mathcal{F}},\underset{\sim}{\mu},\overline{R}_+,\oplus,\odot)$ is a fuzzy pan-space; S is the set of all simple functions on X; $\underset{\sim}{M}$ is the set of all measurable functions on $\underset{\sim}{\mathcal{F}}$ and $\underset{\sim}{M}^+=\{f: f\in\underset{\sim}{M}$ and $f\geqslant 0\}$, $\mathcal{B}=\{E: E\in\underset{\sim}{\mathcal{F}}$, E is a classical set$\}$. (cf. [1])

Definition 1.1 Let $(X,\underset{\sim}{\mathcal{F}},\underset{\sim}{\mu},\overline{R}_+,\oplus,\odot)$ be a fuzzy pan-space, $\underset{\sim}{\mu}$ is called fuzzy pan-additive if and only if for any $\underset{\sim}{A},\underset{\sim}{B}\in\underset{\sim}{\mathcal{F}}$, $\underset{\sim}{A}\cap\underset{\sim}{B}=\emptyset$, we have $\underset{\sim}{\mu}(\underset{\sim}{A}\cup\underset{\sim}{B})=\underset{\sim}{\mu}(\underset{\sim}{A})\oplus\underset{\sim}{\mu}(\underset{\sim}{B})$.

Proposition 1.1 For any $f\in\underset{\sim}{M}^+$, there exists $\{s_n\}\subset S$, such that $s_n \nearrow f$.

Proof. For any natural number n , we define

$$s_n(x)=\bigoplus_{k=0}^{n\cdot 2^n-1}\left[\frac{k}{2^n}\odot\chi_{\{\frac{k}{2^n}\le f(x)<\frac{k+1}{2^n}\}}(x)\right]\oplus\left[n\odot\chi_{\{f(x)\ge n\}}(x)\right]\quad \forall x\in X,$$

then $\{s_n\}$ is the required sequence of simple functions.

Theorem 1.1[1]. For any given $\underset{\sim}{A}\in\underset{\sim}{\mathcal{F}}$, if we define $\mu^*(E)=\underset{\sim}{\mu}(\underset{\sim}{A}\cap E)$ $\forall E\in\mathcal{B}$, then μ^* is a fuzzy measure on $\mathcal{B}$.

Theorem 1.2[1].(Transformation theorem) Suppose that $(X, \underset{\sim}{\mathcal{F}}, \underset{\sim}{\mu})$ is a fuzzy measure space, $\underset{\sim}{A}\in\underset{\sim}{\mathcal{F}}$, $f\in\underset{\sim}{M}^+$, then

$$(p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}=(p)\int_X f d\mu^* .$$

The following discussions will proceed in fuzzy pan-space $(X, \underset{\sim}{\mathcal{F}}, \underset{\sim}{\mu}, \overline{R}_+, \oplus, \odot)$ and we always suppose that $\underset{\sim}{\mu}$ is fuzzy pan-additive.

§2 SOME PROPERTIES OF FUZZY PAN-INTEGRAL

Definition 3.1 The real-valued function on X $s(x)=\bigoplus_{i=1}^{n}(\alpha_i\odot\chi_{E_i}(x))$ is said to be a simple function if $\alpha_i\ge 0$, $E_i\in\mathcal{B}$ $(i=1,2,\cdots,n)$, $E_i\cap E_j=\emptyset$ $(i\ne j)$ and $\bigcup_{i=1}^{n}E_i=X$.

Note: The definition of a simple function here is different from [1]. In [1], to assure $P_{\underset{\sim}{A}}(s)$ uniquely decided by s, we demanded the supplement condition $\alpha_i\ne\alpha_j$ $(i\ne j)$. But when $\underset{\sim}{\mu}$ is fuzzy pan-additive, the supplement condition can be dropped.

Proposition 2.1 For any $\underset{\sim}{A}\in\underset{\sim}{\mathcal{F}}$, $s\in S$, if $s=\bigoplus_{i=1}^{n}(\alpha_i\odot\chi_{E_i})=\bigoplus_{j=1}^{m}(\beta_j\odot\chi_{F_j})$, then

$$P_{\underset{\sim}{A}}(s)=\bigoplus_{i=1}^{n}(\alpha_i\odot\underset{\sim}{\mu}(\underset{\sim}{A}\cap E_i))=\bigoplus_{j=1}^{m}(\beta_j\odot\underset{\sim}{\mu}(\underset{\sim}{A}\cap F_j)) .$$

Proof. By using the fuzzy pan-additivity of $\underset{\sim}{\mu}$, it is easy to prove this conclusion.

From Proposition 2.1, we can obtain the following conclusion.

<u>Proposition 2.2</u> For any $\underset{\sim}{A}\in\underset{\sim}{\mathcal{F}}$, if $s_1, s_2\in S$ and $s_1(x)\leqslant s_2(x)$ $\forall x\in X$, then $P_{\underset{\sim}{A}}(s_1)\leqslant P_{\underset{\sim}{A}}(s_2)$.

So, we know for any $s\in S$, the equality $(p)\int_{\underset{\sim}{A}} s\,d\underset{\sim}{\mu}=P_{\underset{\sim}{A}}(s)$ holds.

Synthesize Proposition 1.1, 2.2 and Theorem 4.1 in [1], we can know that for any $f\in\underset{\sim}{M}^+$, there exists $\{s_n\}\subset S$ such that $s_n\nearrow f$, and $(p)\int_{\underset{\sim}{A}} f\,d\underset{\sim}{\mu}=\lim\limits_{n\to\infty} P_{\underset{\sim}{A}}(s_n)$.

<u>Proposition 2.3</u> Let $\underset{\sim}{A}, \underset{\sim}{B}\in\underset{\sim}{\mathcal{F}}$, $f, f_1, f_2\in\underset{\sim}{M}^+$, then

1). $(p)\int_{\underset{\sim}{A}} C\,d\underset{\sim}{\mu}=C\odot\underset{\sim}{\mu}(\underset{\sim}{A})$, $C\in[0,\infty)$;

2). $(p)\int_{\underset{\sim}{A}\cup\underset{\sim}{B}} f\,d\underset{\sim}{\mu}=(p)\int_{\underset{\sim}{A}} f\,d\underset{\sim}{\mu}\oplus(p)\int_{\underset{\sim}{B}} f\,d\underset{\sim}{\mu}$, where $\underset{\sim}{A}\cap\underset{\sim}{B}=\emptyset$. Particularly if $\underset{\sim}{\mu}(\underset{\sim}{B})=0$, then $(p)\int_{\underset{\sim}{A}\cup\underset{\sim}{B}} f\,d\underset{\sim}{\mu}=(p)\int_{\underset{\sim}{A}} f\,d\underset{\sim}{\mu}$;

3). $(p)\int_{\underset{\sim}{A}}(f_1\oplus f_2)\,d\underset{\sim}{\mu}=(p)\int_{\underset{\sim}{A}} f_1\,d\underset{\sim}{\mu}\oplus(p)\int_{\underset{\sim}{A}} f_2\,d\underset{\sim}{\mu}$;

4). $(p)\int_{\underset{\sim}{A}}(a\odot f)\,d\underset{\sim}{\mu}=a\odot(p)\int_{\underset{\sim}{A}} f\,d\underset{\sim}{\mu}$, $a\in[0,\infty)$.

Proof. 1). By using Transformation Theorem and Theorem 1.1 in [2], we have

$$(p)\int_{\underset{\sim}{A}} C\,d\underset{\sim}{\mu}=(p)\int_X C\,d\mu^*=C\odot\mu^*(X)=C\odot\underset{\sim}{\mu}(\underset{\sim}{A}\cap X)=C\odot\underset{\sim}{\mu}(\underset{\sim}{A}) ;$$

2). For any $E\in\mathcal{B}$, we denote $\mu_1^*(E)=\underset{\sim}{\mu}(\underset{\sim}{A}\cap E)$, $\mu_2^*(E)=\underset{\sim}{\mu}(\underset{\sim}{B}\cap E)$, $\mu^*(E)=\underset{\sim}{\mu}((\underset{\sim}{A}\cup\underset{\sim}{B})\cap E)$.

From the hypothesis that $\underset{\sim}{A}\cap\underset{\sim}{B}=\emptyset$ and the fuzzy pan-additivity of $\underset{\sim}{\mu}$, we can prove $\mu^*(E)=\mu_1^*(E)\oplus\mu_2^*(E)$. Therefore,

$$(p)\int_{\underset{\sim}{A}\cup\underset{\sim}{B}} f\,d\underset{\sim}{\mu}=(p)\int_X f\,d\mu^*=(p)\int_X f\,d(\mu_1^*\oplus\mu_2^*)=$$

$(p)\int_X f d\mu_1^* \oplus (p)\int_X f d\mu_2^* = (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \oplus (p)\int_{\underset{\sim}{B}} f d\underset{\sim}{\mu}$;

3). By using Theorem 1.2 given in [2] and the Transformation theorem, it follows that

$$(p)\int_{\underset{\sim}{A}} (f_1 \oplus f_2) d\underset{\sim}{\mu} = (p)\int_X (f_1 \oplus f_2) d\mu^* =$$
$$(p)\int_X f_1 d\mu^* \oplus (p)\int_X f_2 d\mu^* = (p)\int_{\underset{\sim}{A}} f_1 d\underset{\sim}{\mu} \oplus (p)\int_{\underset{\sim}{A}} f_2 d\underset{\sim}{\mu} ;$$

4). The proof of this conclusion is similar to 3) .

Proposition 2.4 Let $\underset{\sim}{A} \in \mathcal{F}$, $\{f_n, f, g\} \subset \underset{\sim}{M}^+$, then

1). $f \overset{a.e.}{\underset{X}{=}} g \Longrightarrow (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = (p)\int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$;

2). Write $\bigoplus_{n=1}^{\infty} f_n = \lim_{n\to\infty} \bigoplus_{i=1}^{n} f_i$, if $\bigoplus_{n=1}^{\infty} f_n < \infty$, then

$$(p)\int_{\underset{\sim}{A}} \Big(\bigoplus_{n=1}^{\infty} f_n\Big) d\underset{\sim}{\mu} = \bigoplus_{n=1}^{\infty} (p)\int_{\underset{\sim}{A}} f_n d\underset{\sim}{\mu} .$$

Proof. 1). The proof of this conclusion is easily given by using Theorem 1.4 given in [2] and Transformation theorem.

2). Let $f = \bigoplus_{n=1}^{\infty} f_n$, $F_n = \bigoplus_{i=1}^{n} f_i$, then $F_n \nearrow f$. From Theorem 4.1 in [1] and Proposition 2.3, 3), we have

$$\bigoplus_{n=1}^{\infty} (p)\int_{\underset{\sim}{A}} f_n d\underset{\sim}{\mu} = \lim_{n\to\infty} \bigoplus_{i=1}^{n} (p)\int_{\underset{\sim}{A}} f_i d\underset{\sim}{\mu} = \lim_{n\to\infty} (p)\int_{\underset{\sim}{A}} \bigoplus_{i=1}^{n} f_i d\underset{\sim}{\mu} =$$
$$\lim_{n\to\infty} (p)\int_{\underset{\sim}{A}} F_n d\underset{\sim}{\mu} = (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = (p)\int_{\underset{\sim}{A}} \bigoplus_{n=1}^{\infty} f_n d\underset{\sim}{\mu} .$$

The proof of this proposition is complete .

§ 3 CONVERGENCE THEOREMS

Definition 3.1 An ordered commutative semi-ring $(\overline{R}_+, \oplus, \odot)$ is called an additive ideal semi-ring if and only if

$$\bigoplus_{i=1}^{n} ((a_i + b_i) \odot c_i) \leq \Big(\bigoplus_{i=1}^{n} (a_i \odot c_i)\Big) + \Big(\bigoplus_{i=1}^{n} (b_i \odot c_i)\Big)$$
$$\forall\ a_i, b_i, c_i \in \overline{R}_+, \ (i=1,2,\cdots,n) .$$

Evidently, $(\overline{R}_+, \vee, \wedge)$, $(\overline{R}_+, \vee, \cdot)$ and $(\overline{R}_+, +, \cdot)$ are all additive ideal semi-rings.

Definition 3.2 If $\underset{\sim}{\mu}(X) < \infty$, then we call fuzzy pan-space $(X, \underset{\sim}{\mathcal{F}}, \underset{\sim}{\mu}, \overline{R}_+, \oplus, \odot)$ totally finite.

Lemma 3.1 Suppose that $(X, \underset{\sim}{\mathcal{F}}, \underset{\sim}{\mu}, \overline{R}_+, \oplus, \odot)$ is a fuzzy pan-space, $(\overline{R}_+, \oplus, \odot)$ is an additive ideal semi-ring and $f \in \underset{\sim}{M}^+$, $a \in [0, \infty)$, then for any $\underset{\sim}{A} \in \underset{\sim}{\mathcal{F}}$, we have

$$(p)\int_{\underset{\sim}{A}} (f+a)\,d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{A}} f\,d\underset{\sim}{\mu} + a \odot \underset{\sim}{\mu}(\underset{\sim}{A}) .$$

Proof. By Lemma 2.1 given in [2] and Transformation theorem, we have

$$(p)\int_{\underset{\sim}{A}} (f+a)\,d\underset{\sim}{\mu} = (p)\int_X (f+a)\,d\mu^* \leq$$
$$(p)\int_X f\,d\mu^* + a \odot \mu^*(X) = (p)\int_{\underset{\sim}{A}} f\,d\underset{\sim}{\mu} + a \odot \underset{\sim}{\mu}(\underset{\sim}{A}),$$

the lemma is proved .

From this lemma, we can immediately obtain the following lemma:

Lemma 3.2 Suppose that $(\overline{R}_+, \oplus, \odot)$ is an additive ideal semi-ring, f_1, $f_2 \in \underset{\sim}{M}^+$. If $|f_1 - f_2| \leq \varepsilon$ holds on X, then for any $\underset{\sim}{A} \in \underset{\sim}{\mathcal{F}}$, we have

$$\left| (p)\int_{\underset{\sim}{A}} f_1\,d\underset{\sim}{\mu} - (p)\int_{\underset{\sim}{A}} f_2\,d\underset{\sim}{\mu} \right| \leq \varepsilon \odot \underset{\sim}{\mu}(\underset{\sim}{A}) .$$

By using Lemma 3.2, we can obtain the following theorem:

Theorem 3.1 Suppose that $(X, \underset{\sim}{\mathcal{F}}, \underset{\sim}{\mu}, \overline{R}_+, \oplus, \odot)$ is a totally finite fuzzy pan-space, $(\overline{R}_+, \oplus, \odot)$ is an additive ideal semi-ring and $\{f_n, f\} \subset \underset{\sim}{M}^+$. If $\{f_n\}$ converges uniformly to f on X, then for any $\underset{\sim}{A} \in \underset{\sim}{\mathcal{F}}$,

$$\lim_{n\to\infty} (p)\int_{\underset{\sim}{A}} f_n\,d\underset{\sim}{\mu} = (p)\int_{\underset{\sim}{A}} f\,d\underset{\sim}{\mu} .$$

Lemma 3.3 A fuzzy measure $\underset{\sim}{\mu}$ is autocontinuous from above and for any $\underset{\sim}{A}\in\underset{\sim}{\mathcal{F}}$, μ^* is also autocontinuous from above.

Proof. Let $\{\underset{\sim}{A_n}, \underset{\sim}{A}\}\subset\underset{\sim}{\mathcal{F}}$, $\underset{\sim}{A}\cap\underset{\sim}{A_n}=\emptyset$ $(n=1,2,\cdots)$ and $\underset{\sim}{\mu}(\underset{\sim}{A_n})\to 0$, then $\lim\limits_{n\to\infty}\underset{\sim}{\mu}(\underset{\sim}{A}\cup\underset{\sim}{A_n})=\lim\limits_{n\to\infty}(\underset{\sim}{\mu}(\underset{\sim}{A})\oplus\underset{\sim}{\mu}(\underset{\sim}{A_n}))=\underset{\sim}{\mu}(\underset{\sim}{A})\oplus\lim\limits_{n\to\infty}\underset{\sim}{\mu}(\underset{\sim}{A_n})=\underset{\sim}{\mu}(\underset{\sim}{A})$.

That is to say, $\underset{\sim}{\mu}$ is autocontinuous from above. It is very evident that μ^* is also autocontinuous from above.

Theorem 3.2 Suppose that $(X,\underset{\sim}{\mathcal{F}},\underset{\sim}{\mu},\overline{R}_+,\oplus,\odot)$ is a totally finite fuzzy pan-space, $\{f_n,f\}\subset\underset{\sim}{M}^+$ and $(\overline{R}_+,\oplus,\odot)$ is an additive ideal semi-ring. If $f_n\xrightarrow[X]{a.e.}f$ and if $\{f_n\}$ is uniformly bounded, then $\lim\limits_{n\to\infty}(p)\int_{\underset{\sim}{A}}f_n d\underset{\sim}{\mu}=(p)\int_{\underset{\sim}{A}}f d\underset{\sim}{\mu}$ $\forall\underset{\sim}{A}\in\underset{\sim}{\mathcal{F}}$.

Proof. Using Theorem 2.2 in [2], Lemma 3.3 and Transformation throrem, the proof of this theorem is very easy.

Combine Theorem 3.2 with Theorem 4.1 given in [1], we have

Theorem 3.3 Suppose that $(X,\underset{\sim}{\mathcal{F}},\underset{\sim}{\mu},\overline{R}_+,\oplus,\odot)$ is a totally finite fuzzy pan-space, $(\overline{R}_+,\oplus,\odot)$ is an additive ideal semi-ring and $\{f_n,f\}\subset\underset{\sim}{M}^+$. If $\{f_n\}$ is uniformly bounded and $f_n\xrightarrow[X]{a.e.}f$, then $\lim\limits_{n\to\infty}(p)\int_{\underset{\sim}{A}}f_n d\underset{\sim}{\mu}=(p)\int_{\underset{\sim}{A}}f d\underset{\sim}{\mu}$, $\forall\underset{\sim}{A}\in\underset{\sim}{\mathcal{F}}$.

Definition 3.3 Let $\{f_n,f\}\subset\underset{\sim}{M}^+$, we say $\{f_n\}$ is convergence in measure $\underset{\sim}{\mu}$ to f if and only if for any given $\varepsilon>0$, we have $\lim\limits_{n\to\infty}\underset{\sim}{\mu}(|f_n-f|\geq\varepsilon)=0$. We denote it by $f_n\xrightarrow[X]{\mu}f$.

In the final of this paper, we give the principal theorem of this section.

Theorem 3.4 Suppose that $(X,\underset{\sim}{\mathcal{F}},\underset{\sim}{\mu},\overline{R}_+,\oplus,\odot)$ is a totally finite fuzzy pan-space, $(\overline{R}_+,\oplus,\odot)$ is an additive ideal semi-ring and $\{f_n, f\} \subset \underset{\sim}{M}^+$. If $\{f_n\}$ is uniformly bounded and $f_n \xrightarrow[X]{\underset{\sim}{\mu}} f$, then

$$\lim_{n\to\infty} (p)\int_{\underset{\sim}{A}} f_n d\underset{\sim}{\mu} = (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}\ ,\ \forall\, \underset{\sim}{A} \in \underset{\sim}{\mathcal{F}}\ .$$

Proof. By Transformation theorem, we have

$$\lim_{n\to\infty} (p)\int_{\underset{\sim}{A}} f_n d\underset{\sim}{\mu} = \lim_{n\to\infty} (p)\int_X f_n d\mu^*\ .$$

It follows from the hypothesis $f_n \xrightarrow[X]{\underset{\sim}{\mu}} f$ that

$$\mu^*(|f_n - f| \geq \varepsilon) = \underset{\sim}{\mu}(\underset{\sim}{A} \cap \{|f_n - f| \geq \varepsilon\}) \leq \underset{\sim}{\mu}(|f_n - f| \geq \varepsilon) \to 0,\ \forall\, \varepsilon > 0.$$

Therefore, $f_n \xrightarrow[X]{\mu^*} f$, the conditions of Theorem 2.4 in [2] are satisfied, hence $\lim_{n\to\infty} (p)\int_X f_n d\mu^* = (p)\int_X f d\mu^*$.

Observe that $(p)\int_X f d\mu^* = (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}$, the proof of this theorem is complete.

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