A NOTE ON A SUM OF OBSERVABLES IN F-QUANTUM SPACES AND ITS PROPERTIES

Anatolij DVUREČENSKIJ
Mathematical Institute
of the Slovak Academy
of Sciences,
Obrancov mieru 49,
814 73 Bratislava
Czechoslovakia

Anna TIRPÁKOVÁ
Archaeological Institute
of the Slovak Academy
of Sciences,
949 21 Nitra - Hrad,
Czechoslovakia

For an F-quantum system (X,M) we define a sum of any pair of F-observables x and y. We show that the sum always exists, and we use it in order to define some of known convergence theorems of the conventional probability theory.

DEFINITION 1. An F-quantum space is a couple (X,M), where X is a nonempty set and $M \subset [0,1]^X$ such that the followin conditions are satisfied:

(i) if $[\![1]\!]_X(x) = 1$ for any $x \in X$, then $[\![1]\!]_X \in M$;
(ii) if $f \in M$, then $f^{\perp} := 1 - f \in M$;
(iii) if $[\![1/2]\!]_X(x) = 1/2$ for any $x \in X$, then $[\![1/2]\!]_X \notin M$;
(iv) $\bigvee_{n=1}^{\infty} f_n := \sup_n f_n \in M$, for any $\{f_n\}_{n=1}^{\infty} \subset M$.

The system M is called in the fuzzy sets theory a soft σ-algebra (Piasecki [1]).

DEFINITION 2. An F-observable on an F-quantum space (X, M) is a mapping $x: B(R^1) \longrightarrow M$ satisfying the following properties:

(i) $x(A^c) = 1 - x(A)$ for every $A \in B(R^1)$;
(ii) if $\{A_n\}_{n=1}^{\infty} \subset B(R^1)$, then $x(\bigcup_{n=1}^{\infty} A_n) = \bigvee_{n=1}^{\infty} x(A_n)$.

In particular, let $a \in M$, then mapping

$$x_a(E) = \begin{cases} a \wedge a^{\perp} & 0,1 \notin E \\ a^{\perp} & 0 \in E,\ 1 \notin E \\ a & 0 \notin E,\ 1 \in E \\ a \vee a^{\perp} & 0,1 \in E \end{cases} \qquad (E \in B(R^1))$$

is an F-observable of (X,M) called the indicator of the fuzzy set $a \in M$. If $f: R^1 \longrightarrow R^1$ is a Borel measurable function and x is an F-observable, then $f \circ x: E \longrightarrow x(f^{-1}(E))$, $E \in B(R^1))$, is an F-observable, too.

DEFINITION 3. A nonempty subset $A \subset M$ is said to be a Boolean algebra (σ-algebra) of an F-quantum space (X,M) if

(i) there are the minimal and maximal elements 0_A and 1_A from A such that, for any $f \in A$, $0_A \leqslant f \leqslant 1_A$, and $f \vee f^{\perp} = 1_A$;

(ii) A is with respect to $\wedge, \vee, \perp, 0_A, 1_A$ a Boolean algebra (σ-algebra).

It is clear that $0_A \neq 1_A$.

In particular, if x is an F-observable of (X,M), then the range $R(x) = \{x(E): E \in B(R^1)\}$ is a Boolean σ-algebra of (X,M), with the minimal and maximal elements $x(\emptyset)$ and $x(R^1)$, respectively.

DEFINITION 4. We say that two elements $a,b \in M$ are:

(i) orthogonal if $a \leqslant 1 - b$ and we write $a \perp b$;

(ii) compatible if $a = a \wedge b \vee a \wedge b^{\perp}$,
$b = b \wedge a \vee b \wedge a^{\perp}$ and we write $a \leftrightarrow b$;

(iii) strongly compatible if $a \leftrightarrow b \leftrightarrow a^{\perp} \leftrightarrow b^{\perp} \leftrightarrow a$.

Two observables x and y are compatible if $x(E) \longleftrightarrow y(F)$ for any $E,F \in B(R^1)$.

THEOREM 1. (Dvurečenskij, A., Riečan, B. [2]). Let $\{a_t: t \in T\}$ be a system of fuzzy sets from M. The following assertions are equivalent:

(i) $\{a_t: t \in T\}$ is a system of mutually strongly compatible elements;

(ii) $a_t \vee a_t^{\perp} = a_s \vee a_s^{\perp}$ for any $s,t \in T$;

(iii) there is a Boolean σ-algebra of M containing all

$\{a_t: t \in T\}$.

THEOREM 2. Let x be an F-observable of an F-quantum space (X,M) and let $B_x(t) = x((-\infty, t))$, $t \in R^1$. Then the system $\{B_x(t): t \in R^1\}$ fulfils the following conditions:

(i) $B_x(s) \leq B_x(t)$ if $s \leq t$;

(ii) $\bigvee_t B_x(t) = a$;

(iii) $\bigwedge_t B_x(t) = a^\perp$;

(iv) $\bigvee_{t<s} B_x(t) = B_x(s)$;

(v) $B_x(t) \vee B_x^\perp(t) = a$, where $a = x(R^1)$ and $a^\perp = x(\emptyset)$.

Conversely, if a system $\{B_t: t \in R^1\}$ of fuzzy sets of an F-quantum space (X,M) fulfils the conditions (i) - (v) for some $a \in M$, then there is a unique F-observable x such that $B_x(t) = B(t)$ for any t, and $x(R^1) = a$.

PROOF. We prove only the converse implication. Due to (v), the system $\{B(t): t \in T\}$ consists of mutually strongly compatible elements, so that according to Dvurečenskij, A., Riečan, B. [2], there is a Boolean σ-algebra A of M, containing all B(t). Due to the result of Dvurečenskij, A. [3], there exists a unique F-observable x, such that $x((-\infty, t)) = B(t)$.

DEFINITION 5. Let x and y be two F-observables of (X,M). If the system $\{B_{x\oplus y}(t): t \in R^1\}$, where

$$B_{x\oplus y}(t) = \bigvee_{r \in Q} (B_x(r) \wedge B_y(t - r)), \quad t \in R^1,$$

determines an F-observable z of (X,M), then we call it a sum of x and y and we write $z = x \oplus y$. It is clear that if the sum exists, then it is unique.

THEOREM 3. For every two F-observables x and y of an F-quantum space (X,M) there exists its sum.

PROOF. We show that the system $\{B_{x\oplus y}(t): t \in R^1\}$ fulfils the conditions of Theorem 2. For the proof of Theorem 3 is useful the following lemma.

Lemma 1. Let S be a countable dense set in R^1. Let us denote for the observables x,y:

$B^S_{x\oplus y}(t) = \bigvee_{s\in S} (B_x(s) \wedge B_y(t - s))$, then $B^S_{x\oplus y}(t) = B_{x\oplus y}(t)$ for every $t \in R^1$.

Now, the proof of (i), (ii), (iv) is simple, due to the σ-continuity of M. It may be proved that $B_{x\oplus y}(t) \vee B^{\perp}_{x\oplus y}(t) = x(R^1) \wedge y(R^1)$ so that $a = x(R^1) \wedge y(R^1)$. To prove $\bigwedge_t B_{x\oplus y}(t) = x(\emptyset) \vee y(\emptyset) = a^{\perp} = (x(R^1) \wedge y(R^1))^{\perp}$, we take into account property (v) that $\{B_{x\oplus y}(t) : t \in R^1\}$ is a system of mutually strongly compatible elements of an F-quantum space (X,M). According to Theorem 1, there exists a Boolean σ-algebra A containing all $B_{x\oplus y}(t)$ for any $t \in R^1$. Every Boolean σ-algebra A in M is σ-distributive, i.e., if T and S are countable sets, $\{a_{ts} : t \in T, s \in S\} \subset A$, then

$$\bigvee_{t\in T} \bigwedge_{s\in S} a_{ts} = \bigwedge_{\varphi\in S^T} \bigvee_{t\in T} a_{t\varphi(t)}$$

which enables us to prove (iii).

Q.E.D.

PROPERTIES OF THE SUM

(i) $x \oplus y = y \oplus x$ for any two F-observables x,y;

(ii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for any three F-observables x,y and z;

(iii) if $x \leftrightarrow y$ (due to Dvurečenskij, A., Riečan, B.[2], there exists an F-observable z and two Borel measurable functions f and g such that $x = f \circ z$, $y = g \circ z$), then $x \oplus y = (f + g) \circ z$.

Hence, if M consists of crisp subsets, that is, M is a σ-algebra of subsets of X, then the sum of F-observables coincides with a pointwisely defined sum. Ideed, in this case, for x and y, there are unique mappings: $u, v : X \rightarrow R^1$, such that $x(E) = u^{-1}(E)$ and $y(F) = v^{-1}(F)$; $E, F \in B(R^1)$, and $x \oplus y(E) = (u + v)^{-1}(E)$ for any $E \in B(R^1)$.

(iv) Let $a \in R^1$ and put

$$I_a(E) = \begin{cases} [1]_X & a \in E \\ [0]_X & a \notin E, \end{cases}$$

then $x \oplus I_a = f_a \circ x$, where $f_a(t) = t + a$.

(v) We define the subtraction of x and y as follows $x \ominus y = x \oplus (-y)$, where $(-y)(E) = y(\{t: -t \in E\})$, $E \in B(R^1)$.

CONVERGENCES OF F-OBSERVABLES

DEFINITION 6. An F-state on an F-quantum space (X,M) is a mapping $m: M \to [0,1]$ such that

(i) $m(f \vee (1 - f)) = 1$ for every $f \in M$;

(ii) if $f_i \in M$ $(i = 1, 2, \ldots)$ and $f_i \leq 1 - f_j$ $(i \neq j)$ then $m(\bigvee_i f_i) = \sum_i m(f_i)$.

In fuzzy set theory the mapping m is called a P-measure (Piasecki [1]).

DEFINITION 7. Let x be an F-observable, and let m be an F-state. If integral $m(x) := \int_{R^1} t \, dm_x(t)$ exists, then $m(x)$ is called the mean value of x in M, where $m_x: E \to m(x(E))$, $E \in B(R^1)$, is a measure on $B(R^1)$.

DEFINITION 8. We say that a sequence of $\{x_n\}_{n=1}^{\infty}$ of F-observables of (X,M) converges to an F-observable x:

(i) in an F-state m, if for every $\varepsilon > 0$, $\lim m((x_n - x)[-\varepsilon,\varepsilon]) = 1$;

(ii) almost everywhere in an F-state m, if for every $\varepsilon > 0$, $m(\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} (x_n - x)[-\varepsilon,\varepsilon]) = 1$;

(iii) everywhere, if for every $\varepsilon > 0$, $\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} ((x_n - x)[-\varepsilon,\varepsilon]) = 1$;

(iv) uniformly on $a \in M$, if for every $\varepsilon > 0$ $\exists n_0$ such that, for all $n \geq n_0$, $(x_n - x)[-\varepsilon,\varepsilon] \geq a$;

(v) uniformly, if for every $\varepsilon > 0$ there exists n_0 such that, for all $n \geq n_0$, $(x_n - x)[-\varepsilon,\varepsilon] = 1$;

(vi) almost uniformly in an F-state m, if for every $\varepsilon > 0$, there exists an element $a \in M$ such that $m(a^{\perp}) \leq \varepsilon$, and a sequence $\{x_n\}_{n=1}^{\infty}$ converges to x uniformly on a.

Our concept of the sum enables us to formulate and prove many familiar limit theorems known in classical probability theory. For example, laws of large numbers, central limit theorem, Lebesgue convergence theorem, Egorov theorem, ergodic theorem, etc. For example, we formulate Egorov theorem and ergodic theorem:

EGOROV THEOREM. If the sequence $\{x_n\}_{n=1}^{\infty}$ of F-observables converges to an F-observable x in an F-state m, then this sequence converges to an F-observable x almost uniformly in an F-state m.

The Egorov theorem may be proved directly or we define the ideal of zero sets $I_m = \{a: m(a) = 0\}$ and we define the factor system M/I_m which is a Boolean σ-algebra. This method may be used also for the proof of the ergodic theorem.

A mapping $\tau: M \to M$ such that

(i) $\tau(a^{\perp}) = \tau(a)^{\perp}$, $a \in M$;

(ii) $\tau(\bigvee_{i=1}^{\infty} a_i) = \bigvee_{i=1}^{\infty} \tau(a_i)$, $\{a_i\}_{i=1}^{\infty} \subset M$;

is called a homomorphism of M into M. We say that a homomorphism τ of M is ergodic in a state m if

(i) $m(\tau a) = m(a)$, $a \in M$;

(ii) $\tau a = a$ implies $m(a) \in \{0, 1\}$.

ERGODIC THEOREM. Let x be an F-observable, and let τ be a homomorphism of (X,M), ergodic in a state m. Let m(x) = = 0, then $\frac{1}{n} \bigoplus_{i=1}^{n-1} \tau^i \circ x \to I_0$ almost everywhere in m, where I_0 is an observable of (X,M) such that $I_0(E) = \begin{cases} 1, & \text{if } 0 \in E \\ 0, & \text{if } 0 \notin E. \end{cases}$

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